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WARRANTY POLICIES: CONSUMER VALUE

VS. MANUFACTURER COSTS

BY

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CHAPTER 1

WARRANTY POLICIES

1.1. Introduction

A warranty is a contractual obligation incurred by a manufacturer or vendor in connection with the sale of an item or service. The warranty specifies that the manufacturer agrees to remedy certain defects or failures in the commodity sold. The purpose of the warranty is to promote sales by assuring the quality of the item or service to the customer.

There are many different types of warranties but most seem to fall into one of two categories as defined by the Federal Trade Commission. These two categories are the "full warranty" and the "limited warranty". A full warranty specifies that the product must be repaired or replaced within a reasonable time at no charge to the consumer. In a limited warranty the consumer is frequently expected to pay at least a portion of the cost of repairing or replacing the product.

In planning a warranty policy many factors must be taken into consideration. These factors range from the consumers' typical psychological perception that a longer warranty implies a "better" product, to the quantitative analyses that show an additional non-negative cost is incurred (by the manufacturer) whenever a warranty is offered. Increasing consumer awareness (of the value of warranties) will inevitably be another factor in the managerial decision of choosing the type and length of warranty to offer. This advance in "consumerism" will require increasing attention on the part of manufacturers wishing to improve or even maintain their competitive positions.

This study considers most of the more common warranty policies (both stated and unstated) from both the consumer's point of view and from the manufacturer's point of view. It is assumed throughout that all costs are positive, all repairs are instantaneous, and all item lifelengths are nonzero.

1.2. Full-Warranty Policies

Perhaps the most common warranty policy is the simple fixed-time warranty, (henceforth referred to as the standard warranty policy). In this policy, anytime the purchased item fails before time W, the warranty length, the item is either repaired or replaced free of charge. If the new or repaired item also fails within the warranty period it too is repaired or replaced free of charge. This continues until time W after which the consumer must pay the full cost of either repairing or replacing the item. In this policy the consumer is effectively guaranteed that for the original price of the item he will have a functioning item for at least a time period of length W.

A second common warranty policy is the renewing warranty policy. It is frequently offered by the manufacturers of small mechanical and electrical appliances. This is often an unstated policy that works as follows:

For the fixed price P, the consumer buys an item with warranty length W. This warranty is identical to the one previously discussed except that when the item fails (within the warranty period W) the manufacturer not only repairs or replaces the item free of charge, but also gives the consumer a new warranty of length W that supercedes the old warranty. In this way the consumer is guaranteed that for the price of the original item he will receive new or repaired items free of charge until one of the items functions for a time period longer than W.

It is clear that this policy is more expensive from the manufacturers' point of view than the original policy. The way this policy actually arises is typified by the following example.

A consumer purchases a clock-radio from his local Jarco Store. This clock-radio comes with the standard one year warranty (i.e., if it fails within the first year of purchase, bring it back to Jarco and Jarco will repair or replace it free of charge). However, when the clock radio actually fails in 6 months and the consumer returns it to Jarco, rather than going through the complicated process of trying to repair the item, Jarco simply gives the customer a new box which contains a new clock-radio <u>and</u> also includes a new one year warranty.

The two policies above immediately suggest at least two generalizations. The first (Bell - 1961, [3]) is not very useful from a manufacturers' point of view because of the additional paper work required. According to this generalization, if an item is guaranteed for a period of time W and fails at time $x < W$, it is replaced free of charge and the guarantee is extended for a period kx, $0 \leq k \leq 1$. The case $k = 0$ is the standard policy and the $k = 1$ case is the renewing warranty policy.

The second generalization, the (T,W) warranty policy, is one that is currently being used by some manufacturers of small appliances. In this warranty policy the initial item is given a fixed warranty of length W. If the item fails before time $W - T$, $0 \leq T \leq W$, the item is repaired or replaced and the warranty continues unchanged. If the item fails after time $W - T$ but before time W, the item is repaired or replaced and a new warranty of length T is issued.

An equivalent formulation is as follows: If the item fails at time $x < W$, the item is repaired or replaced and a new warranty of length

$$\max (T, W - x)$$

is issued. The policies $T = 0$ and $T = W$ correspond with the standard and renewing warranty policies respectively.

This policy requires very little additional work for the manufacturer since all he needs to do is give a fixed length warranty of length T with any repaired or replaced item (under any warranty). It is up to the consumer to decide whether or not the original warranty is better or worse than the new warranty.

1.3. Limited Warranty Policies

The typical limited warranty is the pro rata warranty. It is found most often when purchasing automobile tires, although lately its popularity has begun to spread to other sectors of transportation. Under the pro rata warranty, if an item fails at time x, $x < W$, then the item is replaced (typically replaced, not repaired), the warranty is renewed, and the customer is charged a fraction x/W of the price of the item. W is the length of the pro rata warranty. The idea behind the pro rata warranty, from the manufacturer's point of view, is simple. Why should the consumer get all that free use out of the item? Thus, the manufacturer "charges" the customer pro rata (from the Latin word for proportional).

Two generalizations of the pro rata warranty have appeared recently on the market. The first of these (Heschel - 1971, [13]) is a pro rata policy with delay. If an item under warranty fails before time s a new item is issued and the warranty is renewed at no charge to the consumer. If the item fails at time x, where $s \leq x \leq W$, a new item is issued, the warranty is renewed, and the customer is charged a fraction $x/W-s - s/W-s$ of the price of the item. x is the failure time of the item, $s \leq x \leq W$, and W is the warranty length. If the item fails after time W the consumer is charged the full price of a new item. The case $s = 0$ corresponds to the original pro rata warranty and the case $s = W$ corresponds to the free replacement renewing warranty policy.

The second pro rata generalization, pro rata with rebate, charges the regular pro rata price from time s on. As before, if the item fails before s, $0 \leq s \leq W$, a new item is issued and a new warranty is given at no charge to the consumer. After time s but before time W, a new item is issued and the warranty is renewed but the consumer is now charged the fraction x/W of the original price. x is once again the failure time of the item and W is the warranty length. This generalization has the same endpoints as the last. If s = 0, it is the original pro rata policy and if s = W, it is the free replacement policy. These three policies are represented graphically below in Figure 1.

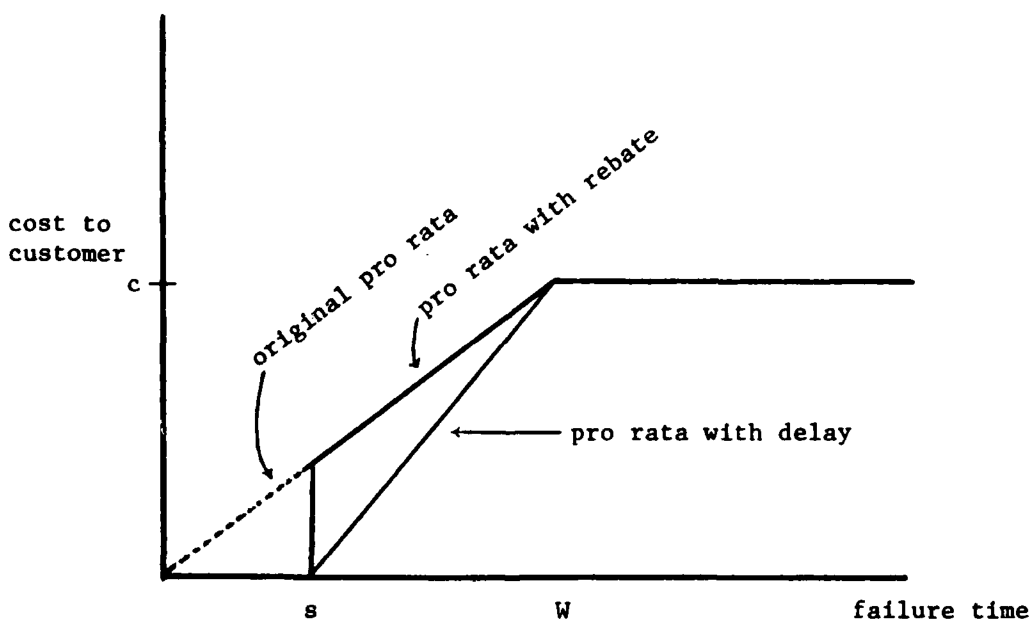


Figure 1.1

Comparison of Pro Rata Warranties

As one can readily see, the second generalization is always superior to the first from the manufacturer's point of view since it always generates at least as much income. This, combined with the fact that from a marketing point of view both generalizations are typically perceived as equivalent (by the average consumer), has led to the almost total abandonment of the first generalization.

Other formulations of generalized warranty policies are, of course, possible. As an example of one that is perhaps too complicated to be used by a manufacturer, one could add an additional $k_1 x$, $0 \leq k_1 \leq 1$ to the warranty period length if the failure time $x \in [s_1, W]$, and charge the consumer a fraction $x/W-s_2 - s_2/W-s_2$ if $x \in [s_2, W]$, (something else otherwise) of the price of the item.

The choice of policy and the selection of the warranty length will depend on production costs, sales price, demand, and distribution of time of failure for both new and repaired items. It will also be different depending upon whether the problem is considered from the point of view of the consumer or the manufacturer.

1.4. History of Warranty Studies

In the last 30 years over 130 articles and technical reports dealing with warranties have been published. The subjects of these articles have varied from advertising to urban transportation and have been published in everything from the Congressional Record to Business Week to the Proceedings of the Reliability and Maintainability Symposiums.

Blischke and Scheuer have done an excellent job of tracking down these sources and compiling them in a single bibliography [7]. They have also classified each of the articles into one or more of eight catagories: structural aspects of warranties, analysis of management decisions with respect to warranties, economic analysis of warranties, related business activities, consumerism, statistical analysis of warranties, legal aspects and miscellaneous. Of the categories pertinent to this study the most important by far is the statistical analysis of warranties. However, the nature of the study also requires consideration of the structural aspects of warranty policies, the analysis of management decisions with respect to warranties and finally, the economic analysis of warranties (along with its supply and demand functions).

The first paper, historically speaking, to deal with warranties from a mathematical or statistical point of view was a Ph.D. dissertation (also published as a technical report) by Lloyd Bell at Stanford University in 1961 [3]. In this dissertation Bell assumes a particular type of demand function and derives conditions under which certain types of warranty policies are optimal. He also includes a short section entitled, "Expected Rate of Profit Per Customer" in which he develops a renewal equation of use in finding "the expected rate of overall profit", a value he wishes to maximize. However, he stops short of solving the renewal equation and instead concentrates on demand functions and their effect on various warranty policies.

No further work relating renewal equations to warranty policies seems to have been done until 1971 when M. S. Heschel published a two-page article entitled "How Much is a Guarantee Worth" [13]. Although Heschel neither explicity mentions nor uses renewal equations in his article he does use the concept of expected cost which ties directly to renewal theory.

Four years later in 1975, Blischke and Scheuer published the first of several papers using renewal theory to analyze both the standard warranty policy (see Section 1.3) and the pro rata warranty policy [4, 5, 6]. In these papers they laid most of the groundwork for comparing different warranty policies from both the consumer and manufacturers' point of view. They, however, stopped short of calculating the actual cost at which the consumer would be indifferent between purchasing an item with a particular warranty as opposed to an item without warranty; instead they estimated these values.

Many other authors have done mathematical studies of warranty policies without the use of renewal theory. Most of these deal with demand functions and market places. Glickman and Berger, for instance, consider displaced log linear demand functions and do a sensitivity analysis of the elasticity assumptions [12]. Other more recent papers have dealt with such topics as incentive contracts (Marshall, 1980 [16]) and imperfect information and alternative market structures (Courville and Hausman, 1979 [9]). These subjects will not be addressed in this study.

CHAPTER 2

BACKGROUND

2.1. Derivation of the Renewal Equation

If a single customer uses a product for a long period of time and replaces each item immediately upon failure, a renewal sequence is generated. This renewal sequence can then be used to develop many values of interest.

Consider a sequence of items issued at times $0, t_1, t_2, \ldots, t_n, \ldots$ $(0 < t_1 < t_2 < \cdots < t_n < \cdots)$ with corresponding lifetimes $x_1, x_2, x_3, \ldots$ such that $t_1 = x_1$, $t_2 = x_1 + x_2, \ldots, t_n = \sum_{i=1}^{n} x_i$. The $\{x_i\}$ are assumed to be independent identically distributed random variables with distribution function $F(t)$, where $F(0) = 0$. They represent the life length of the various items with x_i being the life length of item i. If we let $N(t)$ be the number of items issued up to time t, (not counting the item issued at time 0) i.e.,

$$N(t) = n : t_{n-1} < t \leq t_n$$

then $N(t)$ is a renewal process.

Many values of interest can be found by taking advantage of the fact that $N(t)$ is a renewal process. In the discussion that follows $Y(\cdot)$ may be interpreted as a cost function and $R(\cdot)$ would, therefore, represent cumulative costs. Other interpretations of $Y(\cdot)$ and $R(\cdot)$ will be examined at the end of this section.

Let $Y(\cdot)$ be an arbitrary function mapping $\mathbb{R} \to \mathbb{R}$ and define $Y_j \equiv Y(x_j)$. The $\{Y_j\}$ thus form a sequence of independent identically distributed random variables such that Y_j and x_k are independent for all $j \neq k$. If we let the sequence $\{Y_j\}$ occur at times $\{t_j\}$, respectively, then we have a sequence of values each occurring at the time of issuance of a new item, such that each value, Y_j, is dependent only upon the life length of the previous item, x_j.

If $R_i(T)$ is the sum of all values Y_k obtained during a time period of length T starting at t_i, and $N \geq i$ is determined by $t_N \leq t_i + T < t_{N+1}$, then

$$R_i(T) = \begin{cases} 0 & N = i \text{ (or } x_i > T) \\ Y_{i+1} + Y_{i+2} + \cdots + Y_N & N > i \; (x_i \leq T) \end{cases} .$$

Note that $R_i(T) = Y_{i+1} + R_{i+1}(T - x_{i+1})$ for $x_{i+1} < T$. Thus,

$$E[R_i(T) | x_{i+1} = u] = \begin{cases} E(Y_{i+1} | x_{i+1} = u) + E[R_{i+1}(T - x_{i+1} | x_{i+1} = u)] & u \leq T \\ 0 & u > T \end{cases}$$

$$= \begin{cases} E(Y_{i+1} | x_{i+1} = u) + E(R_{i+1}(T-u)) & u \leq T \\ 0 & u > T . \end{cases}$$

Removing the dependence on u yields

$$E[R_i(T)] = \int_0^T E(Y_{i+1} | x_{i+1} = u) \, dF(u) + \int_0^T E(R_{i+1}(T-u)) \, dF(u) .$$

Let $V_{i+1}(T)$ be defined by

$$V_{i+1}(T) = \int_0^T E(Y_{i+1} | x_{i+1} = u)\, dF(u) \ .$$

Since the quantities $V_{i+1}(T)$ and $E(R_{i+1}(T-u))$ are not dependent on i, the subscript can be dropped yielding

$$E(R(T)) = V(T) + \int_0^T E(R(T-u))\, dF(u) \ . \tag{1}$$

This is the generalized renewal equation discussed in Karlin and Taylor [15]. If $V(T)$ is a bounded function, then there exists one and only one function $E(R(T))$ bounded on finite intervals satisfying (1). This function is

$$E(R(T)) = V(T) + \int_0^T V(T-u)\, dM(u)$$

where $M(u) = \sum_{i=1}^{\infty} F^{(i)}(u) = E(N(u))$ and $F^{(i)}$ is the i-fold convolution of F with itself.

A value of interest is

$$\lim_{t\to\infty} \frac{E(R(t))}{t} \ ,$$

the expected "value per unit time" of the system. To calculate this the following well known lemma (Karlin & Taylor, Chapter 5 [15]) is necessary:

Lemma 2.1. (Blackwell's Theorem) Let F be the distribution function of a positive random variable with mean μ. If F is not arithmetic then $\lim_{t\to\infty} M(t+h) - M(t) = h/\mu$.

Theorem 2.2. If $\hat{R}(t)$ solves the renewal equation

$$\hat{R}(t) = V(t) + \int_0^t \hat{R}(t-u)\, dF(u),$$

where $V(t)$ is a bounded nondecreasing function, then $\lim_{t\to\infty} \frac{\hat{R}(t)}{t} = \frac{V^*}{\mu}$ where $V^* = \lim_{t\to\infty} V(t)$ and $\mu < \infty$ is the mean of $F(u)$.

Proof: By the generalized renewal theorem $\hat{R}(t) = V(t) + \int_0^t V(t-u)\, dM(u)$. Thus,

$$\lim_{t\to\infty} \frac{\hat{R}(t)}{t} = \lim_{t\to\infty} \frac{V(t)}{t} + \lim_{t\to\infty} \int_0^t V(t-u)\, dM(u).$$

By hypothesis $V(t)$ is bounded so $\lim_{t\to\infty} V(t)/t = 0$. Moreover, since $V(t)$ is nondecreasing and tending to V^* for all $\varepsilon > 0$, there exists a T such that $V^* - V(t) < \varepsilon$ for all $t > T$. Splitting the integral into two pieces yields

$$\frac{1}{t}\int_0^t V(t-u)\, dM(u) = \frac{1}{t}\int_0^{t-T} V(t-u)\, dM(u) + \frac{1}{t}\int_{t-T}^{t} V(t-u)\, dM(u) .$$

Note that

$$0 \leq \frac{1}{t} \int_{t-T}^{t} V(t-u)\ dM(u) \leq \frac{V^*}{t} \int_{t-T}^{t} dM(u) = \frac{V^*}{t}[M(t) - M(t-T)]$$

and by the previous lemma

$$\lim_{t\to\infty} \frac{V^*}{t} [M(t) - M(t-T)] = \lim_{t\to\infty} \frac{V^*}{t} \frac{T}{\mu} = 0 .$$

Thus, the second piece of the integral tends to zero. The remaining piece is

$$\frac{1}{t} \int_{0}^{t-T} V(t-u)\ dM(u) .$$

Recall, however, that $t - u \in [T,t]$ which means $V^* - \varepsilon < V(t-u) \leq V^*$ and

$$\frac{(V^*-\varepsilon)}{t} M(t-T) \leq \frac{1}{t} \int_{0}^{t-T} V(t-u)\ dM(u) \leq \frac{V^*}{t} M(t-T) .$$

Taking limits on both sides:

$$\lim_{t\to\infty} \frac{1}{t} \int_{0}^{t-T} V(t-u)\ dM(u) = \frac{V^*}{\mu} . \qquad \boxtimes$$

The above theorem was first proved in 1963 by Johns and Miller [14]. The proof included here is considerably different than the original. Another interesting result occurs if $Y(x)$ is constant for all x.

__Theorem 2.3.__ If $Y(x) \equiv c_0$ then $E(R(t)) = c_0 M(t)$.

__Proof:__ If $Y(x) \equiv c_0$ then

$$V_i(t) = \int_0^t E(Y_i | x_i = u)\, dF(u) = c_0 \int_0^t dF(u) = c_0\, F(t)\ .$$

Plugging into the formula for $E(R(t))$,

$$E(R(t)) = c_0 F(t) + c_0 \int_0^t F(t-u)\, dM(u)$$

$$= c_0 [F(t) + F * M(t)]$$

$$= c_0 M(t)$$

where $F * M(t)$ is the convolution of $F(t)$ and $M(t)$. The last step is a direct result of the fact that

$$M(t) = \sum_{i=1}^{\infty} F^{(i)}(t) = F(t) + \sum_{i=1}^{\infty} F^{(i+1)}(t)\ . \qquad \boxtimes$$

It should be noted further that if $Y_i \equiv c_0$ then $V(t) = c_0 F(t)$ satisfies the hypothesis of Theorem 2.2, i.e., $V(t)$ is a bounded non-decreasing function and, thus, $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{c_0}{\mu}$.

Many useful results can be derived from the two previous theorems. For instance, if $Y_i \equiv 1$ then $R(t)$ is a counting process which counts the number of new items issued in the next t units of time. If Y_i is the cost to the consumer of item i, then $Y_0 + R(t)$ is the total cost to the consumer of starting at time zero and staying in the system until time t. Likewise, if $Y_i = c_i - k_0$ where c_i is the cost to the consumer of item i and k_0 is the fixed cost to the manufacturer of producing item i, then $c_0 - k_0 + R(t)$ is the total profit up to time t. If $Y(x) = x$, another useful result can be found. In this case $Y_j = x_j$, the life length of item j. $R(t)$ is now the sum of the life lengths up to time t not counting the current life length. Hence, $E(t - R(t)) = t - E(R(t))$ is the expected life of the current item thus far, and $\mu + E(R(t))$ is the expected time of the next replacement. $\mu + E(R(t))$ is also the expected life length of the process given no replacements will occur after time t.

In the following sections these results will be applied to the various warranty policies mentioned in Chapter 1. In particular, such quantities as the additional price the consumer should be willing to pay for a given warranty, how much the manufacturer should charge and, under appropriate demand function assumptions, the optimal warranty length a manufacturer should offer, will be derived.

It should also be noted here that the conditions of Theorem 2.2 are <u>not</u> met for all warranty policies mentioned in Chapter 1. In particular, whenever the warranty is not "renewing" (i.e., when the warranty does not necessarily start over at the time of issuance of a new item) then the cost of the item at time t_j is not an exclusive function of x_j, the life length of the previous item. In these cases a separate analysis will be performed to determine the desired results.

2.2. Notation

The following notation will be used throughout this paper:

x_i life length of item i (assumed iid).

$F(\cdot)$ the distribution function of x_i.

$F(x) = P\{x_i \le x\} \quad i = 1, 2, \ldots$.

It is assumed that $F(0) = 0$ signifying the x_i are positive random variables.

μ the expected value of x_i, $E(x_i) = \int_0^\infty x dF(x)$.

μ_A $\int_0^A x dF(x)$.

t_i $\sum_{j=1}^{i} x_j$. The waiting time until the occurrence of the i^{th} event ($t_0 = 0$ by convention).

$F^{(i)}(t)$ $P\{t_i \le t\}$. The convolution of F with itself i times ($F^{(0)}(t) = 1_{\{t \le 0\}}$ by convention).

$N(t)$ $n : t_{n-1} < t \le t_n$. The counting process.

$M(t)$ $E[N(t)]$. The renewal function.

W the warranty length. In this paper the warranty length is defined to be the minimum length of time W such that if the initial item fails on or after time W, the entire price of a new item must be payed to replace the original.

c_0 the price the consumer must pay to purchase the item (or service).

k_0	the cost to the manufacturer of producing an item (or service).
L	the length of time (predetermined) after which the consumer will no longer pay for a new item. L is assumed for convenience to be greater than or equal to W.
$C(L)$	the total expected cost to the consumer over the period $[0,L]$.
$K(L)$	the total expected cost to the manufacturer over the period $[0,L]$.
$P(L)$	$C(L) - K(L)$. The total expected profit to the manufacturer over the period $[0,L]$.
$\tau(L)$	the expected process length. The first time after L that the item fails and the consumer would have to pay a positive amount to replace the item.
τ	$\tau(0)$.
T	the minimum length of warranty offered with a replaced item (used exclusively in the (T,W) warranty policy - see Chapter 4).
s	the initial "free replacement" time under a pro rata warranty (see Figure 1.1).
$Y(\cdot),R(\cdot)$	dummy functions used for various purposes (as is deemed appropriate). Defined in Section 2.1.

Using the variables defined above one can generate the following values which are of interest in comparing warranty policies.

$\frac{C(L)}{L}$ The cost per unit time to the consumer of staying in the system exactly L units of time.

$\frac{C(L)}{\tau(L)}$ The cost per unit of "useful life" to the consumer (note the definition of L).

$\frac{C(0)}{\tau(0)}$ $= \frac{C_0}{\tau}$. This value is of use in the <u>nonrenewing</u> warranties. In these cases it is the cost per "cycle" of the process, where the cycle times represent regeneration times of the process.

C^* $\lim_{L\to\infty} \frac{C(L)}{\tau(L)}$. The long term average cost per unit of useful time to the consumer.

P^* $\lim_{L\to\infty} \frac{P(L)}{\tau(L)}$. The long term average profit per unit time.

The quantity $C(L)/\tau(L)$ arises naturally when a customer purchases an item with the intention of using either that item or an appropriate replacement of that item until time L. Afterwards, the customer is willing to take a new item for free but is not willing to pay for a replacement.

2.3. <u>Comparing Warranty Policies</u>

There is some question as to the proper methodology to use in assessing different warranty policies. Blischke and Scheuer [4, 5, 6] have suggested comparing total cost through time t, $C(t)$, from the consumer's point of view and profit through time t, $P(t)$, from the manufacturer's point of view. In making these comparisons they assume the customer will stay in the system for exactly t units of time. By making this assumption they have decided that even if the consumer is entitled to a new item, free of charge, after time t, the consumer will refuse it and, in fact, the consumer isn't even interested in whether or not his or her item works at all after time t. There are instances where this is a legitimate assumption but, in general, it seems unrealistic.

A more appropriate assumption is that after time t the customer will accept any free replacements but will not pay any positive amount for a replacement. Note that this new assumption does not change the value of $C(L)$ since the customer will not pay any more after time L. It does, however, change the total expected cost to the manufacturer and, hence, the total expected profits.

An alternate method for comparing warranty policies, from the consumer point of view (that takes the above assumption into account), is to examine $\frac{C(L)}{\tau(L)}$ for the various policies. This term, as defined in the previous section, represents the total expected cost up to time L divided by the total expected "useful life" or, the cost per unit of useful life to the consumer. It has the advantage of taking into account that the consumer will have a functioning item for longer than L.

It should be pointed out that $\tau(L)$ is not necessarily finite.

Example 2.3.1. Consider a renewing warranty policy with warranty length W equal to 2 years. If the item lifelengths x_i are distributed deterministically such that $x_i \equiv 1\frac{1}{2}$ years for all i, then each item will fail after exactly $1\frac{1}{2}$ years use. Since a new warranty of length 2 years is issued with each new item (by definition of a renewing warranty policy), the customer will never have to pay for a new item and

$$\tau(0) = \tau(L) = \infty \; .$$

An unfortunate difficulty involved in this comparison is that $\tau(L)$ is frequently too complicated to calculate. In fact, the only policies (in this report) for which it can explicitly be calculated are those for which a new warranty is issued with every new item. It can be found in these instances because the times of issuance form a renewal sequence.

Whenever the consumer is charged some positive amount for any new item, the evaluation of $\tau(L)$ is simplified tremendously. In these cases there is no need to be concerned with the customer "staying around to get a free item" and $\tau(L)$ can be calculated by Wald's Lemma to be

$$\tau(L) = E\left(\sum_{i=1}^{N(L)+1} x_i\right) = E(N(L)+1)\; E(x_i) = \mu(1+M(L)) \; .$$

By definition the above value is the expected time of the first failure after time L.

<u>Lemma 2.3.2</u>.

$$L \leq \mu(1+M(L)) \leq \tau(L) \; .$$

Proof: By Wald's Lemma, $\mu(1+M(L))$ is the expected time of the next failure after time L and, hence, is greater than or equal to L. $\tau(L)$ is the expected time of the next failure after L at which the consumer must pay some positive amount to have the item replaced. Since $\tau(L)$ requires at least a failure, $\tau(L) \geq \mu(1+M(L))$.

☒

For the generalized pro rata policies (the pro rata with delay and the pro rata with rebate), the time after which the consumer must pay for a new item, s, is typically small. If $F(s)$ is likewise small, then the probability of the current item (at time L) having a total lifelength less than or equal to s, $P\{x_{N(L)+1} < s\}$, is very small and $\tau(L)$ can be approximated by $\mu(1+M(L))$. This can frequently be a very useful approximation (especially when compared to the complicated expressions derived in Section 3.4 and Section 3.5).

This approximation is also better than one might expect due to the fact that conditioning upon a particular time frequently causes the expected lifelength of that item to be larger than μ, the expected life length of an arbitrary item. The classic example of this arises for the exponential distribution.

Example 2.3.2. Let $\{x_i\}$ be distributed exponentially with parameter λ. Then $E(x_i) = \mu = \frac{1}{\lambda}$. However, $E(x_{N(L)+1})$

$$E(x_{N(L)+1}) = E(\delta(L)) + F(\gamma(L))$$

where $\delta(L)$ and $\gamma(L)$ are the age and remaining life of the item in use at time L as defined in Barlow and Proschan [2]. The exponential process is memoryless and, hence, $E(\gamma(L)) = \mu$. The expected age of the current item is found by noting

$$E(\delta(L)) = \int_0^\infty P\{\delta(L) > x\}dx$$

$$= \int_0^L P\{\delta(L) > x\}dx$$

$$= \int_0^L e^{-\lambda x}\, dx$$

$$= \mu(1-e^{-\lambda L}) .$$

Thus, $E(x_{N(L)+1}) = 2\mu - \mu e^{-\lambda L}$ and as $L \to \infty$ the expected lifelength of the current item is 2μ or twice the "usual expected life length".

For the other warranty policies discussed in Chapter 1, i.e., the nonpro rata policies, $\tau(L)$ is not so easily estimated. However, these cases all share important features. Whenever the customer must pay for a new item, the cost is exactly c_0 and the warranty renews itself or starts over. Hence, the times when a payment occurs form a sequence of stopping times $s_0 = 0, s_1, s_2, \ldots$ such that $S = \{s_n; n \in I^+\}$ is a renewal process with $Z_i = s_i - s_{i-1}$ as the independent identically distributed random variable that corresponds to the times between regeneration points. Let $G(Z)$ be the corresponding distribution function of the Z_i and let $M_G(t) = E(N_G(t))$ be the appropriate renewal function $[G(0) = 0$ since $F(0) = 0]$.

By definition $\tau = \tau(0) = E(Z_1)$, the expected time of the first payment (not counting the payment at time zero). Since S is a renewal process, Wald's Lemma can once again be used to estimate the time at which the next renewal after L will occur:

$$\tau(L) = E\left(\sum_{i=1}^{N_G(L)+1} Z_i\right) = E(N_G(L)+1)\ E(Z_1) = (M_G(L)+1)\tau\ .$$

Payments still occur at the same points as in the original process and are assumed to be of the same constant value c_0. This makes Theorem 2.3 applicable and $C(L) = C_G(L) = c_0(1+M_G(L))$. Likewise,

$$\frac{C(L)}{\tau(L)} = \frac{C_G(L)}{\tau(L)} = \frac{c_0(1+M_G(L))}{\tau(0)(1+M_G(L))} = \frac{c_0}{\tau}$$

for all L. Thus, for each of the other warranty policies discussed in this paper $\frac{C(L)}{\tau(L)} = \frac{c_0}{\tau}$ and only the value τ need be derived to compare policies from the consumer point of view.

From the manufacturer's point of view, the one quantity which stands out when comparing warranty policies is the profit per customer. Profit per item sold does not work nearly as well due to the varying prices at which items are sold under pro rata warranties. Profit per customer does have the disadvantage of not taking into account the fact that demand is a function of both price and the warranty policy offered. However, once the manufacturer knows the extra price he must charge for a warranty, he can calculate (or estimate) the demand as a function of both the warranty and the price and, hence, optimize over the set of possible warranties.

It is clear that $P(L)$ may not represent a practical quantity to the manufacturer due to the additional costs that he may incur after time L. $P_G(L)$ represents the correct quantity and is found from

$$P_G(L) = C_G(L) - K_G(L) .$$

$K_G(L)$ can be found by noting that

$$K_G(L) = E\left[\sum_{i=1}^{N_G(L)+1} k_i\right]$$

where k_i is the cost incurred by the manufacturer during the i^{th} interval of the regenerative process S. The k_i are independent identically distributed random variables (that are functions of the $t_j \in [s_{i-1}, s_i]$). Applying Wald's Lemma once again yields

$$K_G(L) = E\left[\sum_{i=1}^{N_G(L)+1} k_i\right] = E[k_1](M_G(L)+1)$$

and in particular $P_G(L) = C_G(L) - E[k_1](M_G(L)+1)$.

Recall that in the <u>nonpro rata</u> policies $C_G(L) = c_0(M_G(L)+1)$ and thus, $P_G(L) = [c_0 - E(k_1)](M_G(L)+1)$. In these cases the expected profit per customer per unit time is found to be

$$\frac{P_G(L)}{\tau(L)} = \frac{c_0 - E(k_1)}{\tau} .$$

Fortunately, this is a value that can be calculated, unlike $P_G(L)$ which typically cannot be due to the difficulty in finding $M_G(L)$ [see Blitschke & Scheuer 3].

For the standard pro rata warranty $P_G(L) = P(L) = C(L) - K(L)$ and $K(L)$ is found by $K(L) = k_0(1+M(L))$.

For the two generalizations this analysis becomes more difficult due to the possibility of "free items". However, if the free period $(0,S)$ is small compared to W and $P(S)$ is also small, then $K(L)$ can be used as an approximation for $K_G(L)$ without much loss. In this case the expected profit per consumer per unit time is approximated by

$$\frac{P(L)}{L} = \frac{C(L)-k_0(1+M(L))}{L} .$$

To summarize, the expected cost per unit of useful life and the expected profit per customer per unit time will be used to compare different warranty policies from the consumer's and manufacturer's point of view respectively. For the generalized pro rata policies appropriate approximations of these values can be used. These approximations take into account the fact that the initial price of an item under a pro rata warranty may be considerably larger than the expected cost at any later time. This will be discussed further in Chapter 3.

CHAPTER 3

RENEWING WARRANTY POLICIES

3.1. The Base Policy (No Warranty)

The first policy to be considered is the basic no warranty policy. This policy is an important base to work from because using it one can determine the additional cost the consumer should be willing to pay for a particular warranty, or the additional price the manufacturer should charge to "break even".

Using the notation of Chapter 2, assume an item is sold without warranty for a fixed price c_0, $c_0 > 0$, at time $t_0 = 0$. In this policy each time an item fails before time L it is assumed the customer immediately replaces the item at a charge c_0. The cost per item, to the consumer, satisfies the hypothesis of Theorem 2.3. So, by Theorem 2.3 the expected cost to the consumer in $[0,L]$ is

$$C(L) = c_0 + c_0 M(L)$$

where, as before, $M(t)$ is the renewal function. Assuming each item costs a fixed amount k_0 to produce, $P(L)$ is likewise found to be $P(L) = (c_0-k_0) + (c_0-k_0)\, M(L)$. The expected lifelength of the process, $\tau(L)$, is found by Wald's Lemma:

$$\tau(L) = E\left(\sum_{i=1}^{N(L)+1} x_i\right) = E[N(L)+1]\ E(x_1) = (M(L)+1)\mu\ .$$

The expected cost and profit per unit of useful life are thus

$$\frac{C(L)}{\tau(L)} = \frac{c_0(1+M(L))}{\mu(1+M(L))} = \frac{c_0}{\mu}$$

$$\frac{P(L)}{\tau(L)} = \frac{(c_0-k_0)(1+M(L))}{\mu(1+M(L))} = \frac{c_0-k_0}{\mu}$$

independent of L.

Note that $\lim_{L\to\infty} \frac{C(L)}{L}$ also equals $\frac{c_0}{\mu}$ and $\lim_{L\to\infty} \frac{P(L)}{L} = \frac{c_0-k_0}{\mu}$.

These results come from Lemma 2.2.

Of all the warranty policies discussed only the pro rata warranties and the "renewing warranty" policies have the feature that the cost at time t_i of replacing the item that just failed is independent of x_j for all $j \neq i$. The other policies will be discussed in detail later.

3.2. The "Renewing Warranty" Policy

The renewing warranty policy is so named because a new warranty of equal length to the original is automatically given with the issuance of each new item. The warranty is assumed to be for a fixed length of time W. If the item fails before W a new item is issued free of charge complete with a new warranty. If the item fails after time W, then the item and warranty are replaced for a fixed charge c.

To find $C(L)$, the expected cost to the consumer of replacing the item until time L, the following cost function $Y(\cdot)$ is appropriate

$$Y(x) = \begin{cases} 0 & x \leq W \\ c & x > W \end{cases} .$$

In this case $Y(x)$ represents the cost to the consumer of replacing an item that has lasted exactly x units of time. $R(T)$ is then the sum of the costs up until time T and $E[R(T)]$ satisfies the renewal equation

$$E[R(T)] = \int_0^T [Y(u)]dF(u) + \int_0^T E[R(T-u)]dF(u) .$$

Note that by the definition of Y

$$\int_0^T [Y(u)]dF(u) = \int_0^W 0 \cdot dF(u) + \int_W^T cdF(u)$$

$$= \begin{cases} 0 & T < W \\ c[F(T)-F(W)] & T \geq W \end{cases} .$$

We can safely assume that $T \geq W$ for the first term from the renewal equation [2.1]. However, the second term involves $c[F(T-u)-F(W)]$ and it would be incorrect to assume $T - u \geq W \quad \forall u \in [0,T]$. Applying the renewal equation and noting the above yields

$$E[R(T)] = c[F(T)-F(W)] + \int_0^{T-W} c[F(T-u)-F(W)]dM(u)$$

$$= c[F(T)-F(W)] + \int_0^{T} c[F(T-u)-F(W)]dM(u)$$

$$- \int_{T-W}^{T} c[F(T-u)-F(W)]dM(u)$$

$$= cM(T) - cF(W) - cF(W)M(T) + cF(W)M(T) - cF(W)M(T-W)$$

$$- \int_{T-W}^{T} cF(T-u)dM(u) \ .$$

Since $R(T)$ does not include the cost of the initial item,

$$C(L) = c\left[1+M(L)-F(W) - F(W)M(L-W) - \int_{L-W}^{L} F(L-u)dM(u)\right] .$$

This formula is interesting for a couple of reasons. First of all, it is not the same as the formula derived in [4]. The difference is that in [4] Blitschke and Scheuer simply multiply the expected cost $c(1-F(W))$ by the expected number of renewals $M(L)$. Their technique overestimates the expected cost because in actuality there is no cost in the tails, past time L. Secondly, the two formulas are surprisingly similar and, in fact, if one estimates the final term by assuming $F(T-u) \equiv F(W)$ $\forall$ $u \in [L-W,L]$ then

$$\int_{L-W}^{L} F(t-u)dM(u) \simeq \int_{L-W}^{L} F(W)dM(u)$$

$$= F(W)M(L) - F(W)M(L-W)$$

and

$$C(L) \approx c[1+M(L)-F(W)-F(W)M(L)]$$

$$= c + c(1-F(W))M(L)$$

which is the result found in [1]. In either case the limiting value is

$$c^* = \lim_{L\to\infty} \frac{C(L)}{L} = \frac{c(1-F(W))}{\mu} \quad .$$

The result follows directly from the fact that $F(\cdot)$ is bounded and Theorem 2.2. $V(T)$ in this case is $c[F(T)-F(W)]$ and $\lim_{T\to\infty} V(T) = c[1-F(W)]$.

$K(t)$ can now easily be calculated since the total expected cost to the manufacturer is simply the cost per item times the expected number of items or $k_0(1+M(t))$. Thus, the manufacturer's expected profit in $[0,L]$ is $C(L) - k_0(1+M(L))$. As mentioned before [Section 2.3], this does not reflect the typical expected cost to the manufacturer due to the additional obligations the manufacturer may have incurred at time L. To take these into account the values $\tau = \tau(0)$ and $E(k_1)$ must be found.

τ, the expected time until the consumer must pay for a new item, is easily found by conditioning upon the first failure time and noting that if $x_1 < W$ then the expected time is $x_1 + \tau$. The formula is

$$\tau = \int_0^W (x+\tau)\, dF(x) + \int_W^\infty x dF(x)$$

$$= \int_0^\infty x dF(x) + \tau F(W)$$

$$= \mu + \tau F(W) .$$

Solving for τ,

$$\tau = \mu/(1-F(W)) = \mu/\bar{F}(W) \qquad \text{(by convention } \bar{F}(W) = 1 - F(W)) .$$

A natural question at this point is whether or not $\frac{\mu}{\bar{F}(W)} \geq W$ for all $F(\cdot)$.

<u>Lemma 3.2.1</u>.

$$\frac{\mu}{\bar{F}(W)} \geq W .$$

<u>Proof</u>:

$$\mu = \int_0^\infty x dF(x) \geq \int_0^W x dF(x) + \int_W^\infty W dF(x)$$

$$= \int_0^W x dF(x) + W(1-F(W))$$

$$\geq W\bar{F}(W) .$$

Multiplying both sides through by $\frac{1}{\bar{F}(W)}$ the above result is derived.

$\boxtimes$

To find the expected cost to the manufacturer until the consumer must purchase a new item, a similar analysis is performed.

$$E[k_1] = \int_0^W (k_0 + E(k_1))\ dF(x) + \int_W^\infty k_0\ dF(x)$$

$$= (k_0 + E(k_1))\ F(W) + k_0(1 - F(W))\ .$$

Solving for $E(k_1)$, the expected cost to the manufacturer

$$E(k_1) = \frac{k_0}{\bar{F}(W)}\ ,$$

and the expected profit per customer per unit time is

$$\frac{c_0 - \frac{k_0}{\bar{F}(W)}}{\mu/\bar{F}(W)} = \frac{c_0\bar{F}(W)-k_0}{\mu}\ .$$

The renewing warranty policy has the unique feature that $M_G(t)$, the renewal function for the regenerative process S (as defined in Section 2.3), can be explicitly calculated as a function of W, $F(t)$ and $M(t)$. The analysis can be done directly or by noting that $C(L) = c_0(1+M_G(L))$ since no payments are made after time L. Thus,

$$M_G(L) = M(L) - F(W) - F(W)M(L-W) - \int_{L-W}^{L} F(L-u)dM(u) .$$

$\tau(L)$ can now also be evaluated as

$$\tau(L) = \tau(M_G(L)+1) \qquad \text{and}$$

$$\frac{C(L)}{\tau(L)} = \frac{c_0}{\tau} \qquad \text{for all } L .$$

Example 3.2. If $\{x_i\}$ are distributed exponentially with parameter λ, then

$$M_G(L) = \lambda L - (1-e^{-\lambda W}) - (1-e^{-\lambda W})\lambda(L-W) - \int_{L-W}^{L} [1-e^{-\lambda(L-u)}]\lambda du$$

$$= \lambda L - 1 + e^{-\lambda W} - \lambda L + \lambda W + \lambda Le^{-\lambda W} - \lambda We^{-\lambda W} - \lambda W + 1 - e^{-\lambda W}$$

$$= e^{-\lambda W}[\lambda L-\lambda W] \qquad \text{for } L \geq W .$$

As expected $M_G(L)$ is linear in L (when L is greater than W) for the exponential case.

$C(L)$, the expected cost to the consumer is

$$C(L) = c_0(1+M_G(L)) = c_0 + c_0\, e^{-\lambda W}\, (\lambda L-\lambda W) .$$

$\tau(L)$, the expected useful life is

$$\tau(L) = \frac{\mu}{\bar{F}(W)}\, (1+M_G(L)) = \mu e^{\lambda W} + L - W .$$

For small W

$$e^{\lambda W} \simeq 1 + \lambda W + \frac{(\lambda W)^2}{2} \simeq 1 + \lambda W$$

and

$$\tau(L) \simeq \mu + W + L - W = \mu + L .$$

Likewise, the expected cost to the manufacturer is

$$K(L) = k_0 \, e^{\lambda W} + k_0(\lambda L - \lambda W)$$

which for small W is approximated by $k_0(1+\lambda L)$, the standard no-warranty value.

Comparison 3.2. The price at which the consumer is indifferent between purchasing an item with a renewing warranty and purchasing an item (for c_0) with no warranty is found by comparing $\frac{c_0}{\tau}$ for the two policies. c_0^*, the indifference price, is

$$c_0^* = \frac{c_0}{\mu}\left[\frac{\mu}{\bar{\bar{F}}(W)}\right] = \frac{c_0}{\bar{\bar{F}}(W)} .$$

From the manufacturer's point of view, the total profit under a no-warranty system would be $(c_0 - k_0)(1+M(L))$. Under the renewing warranty system the total profit is

$$\hat{c}_0(1+M_G(L)) - \frac{k_0}{\bar{\bar{F}}(W)}(1+M_G(L))$$

where $\hat{c}_0$ is the price the consumer is charged under the warranty policy. Equating these two equations and solving for $\hat{c}_0$,

$$\hat{c}_0 = (c_0-k_0)\,\frac{(1+M(L))}{(1+M_G(L))} + \frac{k_0}{\bar{F}(W)} \ .$$

Thus, the manufacturer is indifferent between selling a product for $\hat{c}_0$ with a renewing warranty or selling a product for c_0 without warranty.

If the company is interested instead in profit per unit time, then the appropriate measures are

$$\frac{c_0-k_0}{\mu} \quad \text{and} \quad \frac{\hat{c}_0\bar{F}(W)-k_0}{\mu} \ .$$

Not surprisingly, these yield the same value for $\hat{c}_0$ as does the consumer's point of view.

<u>Example 3.2. (continued)</u>. In the exponential case the price at which the manufacturer is indifferent is

$$\hat{c}_0 = \frac{(c_0-k_0)(1+\lambda L)}{e^{-\lambda W}(\lambda L-\lambda W)} + k_0 e^{\lambda W}$$

$$= e^{\lambda W}\left[\frac{c_0\mu}{L-W} + \frac{(c_0-k_0)L}{L-W}\right]$$

and the price at which the consumer is indifferent is $c_0^* = c_0 e^{\lambda W}$.

3.3. The Pro Rata Warranty.

In the standard pro rata warranty policy the consumer is charged a pro rata amount for a replacement item. Figure 3.1 shows the amount the consumer must pay for a new item as a function of the life length of the previous item.

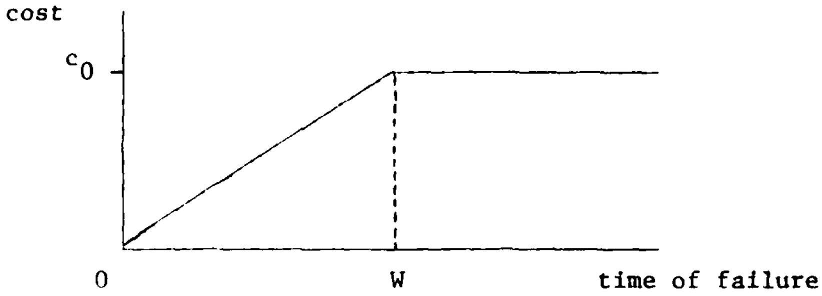


Figure 3.1. Replacement Cost Pro Rata Warranty

Note that the consumer will pay a positive amount any time the item fails. This is because $F(0) = 0$. This means that the consumer never receives a free item and, hence, $\tau(L)$, the expected process length, is found as in Section 2.3 by $\tau(L) = \mu(1+M(L))$.

The total expected cost to the manufacturer is likewise found to be $K(L) = k_0(1+M(L))$.

The total expected cost to the consumer in the period $[0,L]$ is calculated by using the results of Section 2.1. In this case $Y(x)$ is the cost function defined by Figure 3.1.

$$Y(x) = \begin{cases} \frac{c_0 x}{W} & x \le W \\ c_0 & x \ge W \end{cases}$$

and

$$V(T) = \int_0^T E(Y(x) \mid x = u)\, dF(u)$$

$$= \begin{cases} \int_0^T \frac{c_0 u}{W}\, dF(u) & T \le W \\ \int_0^W \frac{c_0 u}{W}\, dF(u) + \int_W^T c_0 dF(u) & T > W \end{cases}$$

$$= \begin{cases} \frac{c_0}{W}\, \mu_T & T \le W \\ \frac{c_0}{W}\, \mu_W + c_0(F(T)-F(W)) & T > W \end{cases}$$

where μ_A is defined to be

$$\mu_A \equiv \int_0^A x dF(x) \ .$$

$E(R(L))$, the expected sum of the costs up until time L, satisfies the renewal equation

$$E[R(L)] = V(L) + \int_0^L E[R(L-u)]\, dF(u) \ .$$

Hence,

$$E[R(L)] = V(L) + \int_0^L V(L-u)\, dM(u)$$

$$= \frac{c_0}{W}\mu_W + c_0(F(L)-F(W))$$

$$+ \int_0^{L-W} \frac{c_0}{W}\mu_W + c_0(F(L-u)-F(W))\, dM(u)$$

$$+ \int_{L-W}^{L} \frac{c_0}{W}\mu_{L-u}\, dM(u) \ .$$

Simplifying, we get

$$E[R(L)] = c_0\, M(L) + c_0\left[\frac{\mu_W}{W} - F(W)\right] + c_0\left[\frac{\mu_W}{W} - F(W)\right] M(L-W)$$

$$+ \int_{L-W}^{L} c_0\left[\frac{\mu_{L-u}}{W} - F(L-u)\right] dM(u) \ .$$

The expected cost to the consumer up until time L is found by adding the initial cost c_0 to the above value:

$$C(L) = c_0(1+M(L)) + c_0\left[\frac{\mu_W}{W} - F(W)\right][1+M(L-W)]$$

$$+ \int_{L-W}^{L} c_0\left[\frac{\mu_{L-u}}{W} - F(L-u)\right] dM(u) \ .$$

Since $V(L)$ satisfies the hypothesis of Theorem 2.2, $\lim_{L\to\infty} \frac{C(L)}{L} = \frac{V^*}{\mu}$, where $V^* = \lim_{t\to\infty} V(t)$.

Thus,

$$\lim_{L\to\infty} \frac{C(L)}{L} = \frac{c_0 \mu_W}{W\mu} + \frac{c_0(1-F(W))}{\mu}$$

$$= \frac{c_0}{W\mu} \int_0^W x dF(x) + \frac{c_0(1-F(W))}{\mu} .$$

The long term average cost per unit of useful time (to the consumer), c^* can now be calculated.

$$c^* = \lim_{L\to\infty} \frac{C(L)}{\tau(L)} = \lim_{L\to\infty} \frac{C(L)}{L} \cdot \frac{L}{\tau(L)} .$$

Both $\frac{C(L)}{L}$ and $\frac{L}{\tau(L)}$ are bounded and positive so the limit can be taken separately over the two pieces yielding

$$c^* = \frac{c_0 \mu_W}{W\mu} + \frac{c_0 \bar{F}(W)}{\mu} .$$

If $W = 0$ in the pro rata policy then in effect one has the base no-warranty-policy. Notice that in this event $C(L)$ simplifies to $c_0(1+M(L))$, the identical formula derived in Section 3.1.

The following example calculates the above values for the exponential case.

<u>Example 3.3</u>. If $F(t) = 1 - e^{-\lambda t}$ then $M(t) = \lambda t$. $\tau(L)$, the expected process length is

$$\tau(L) = \mu(1+\lambda L) .$$

The total expected cost to the manufacturer is

$$K(L) = k_0(1+\lambda L) .$$

C(L), the total expected cost to the consumer is

$$C(L) = c_0(1+M(L)) + c_0\left[\frac{\mu_W}{W} - F(W)\right][1+M(L-W)]$$

$$+ \int_{L-W}^{L} c_0 \frac{\mu_{L-u}}{W} \lambda du - \int_{L-W}^{L} c_0 F(L-u)\lambda du .$$

Using the method of divide and conquer

$$\int_{L-W}^{L} c_0 F(L-u) \lambda du = c_0 \int_{L-W}^{L} [1-e^{-\lambda(L-u)}] \lambda du$$

$$= c_0 \lambda W - c_0 e^{-\lambda L} \int_{L-W}^{L} \lambda e^{\lambda u} du$$

$$= c_0 \lambda W - c_0 e^{-\lambda L} [e^{\lambda} - e^{\lambda L-\lambda W}]$$

$$= c_0 \lambda W - c_0(1-e^{-\lambda W}) .$$

$$\int_{L-W}^{L} c_0 \frac{\mu_{L-u}}{W} \lambda du = \int_{L-W}^{L} c_0 \frac{\int_0^{L-u} x dF(x)}{W} \lambda du$$

$$= \int_{L-W}^{L} \frac{c_0}{W} \left[\mu - \mu e^{-\lambda(L-u)} - (L-u)\ e^{-\lambda(L-u)}\right] \lambda du$$

$$= c_0 - \frac{c_0}{W} \mu e^{-\lambda L} \int_{L-W}^{L} \lambda e^{\lambda u}\ du - \frac{Lc_0}{W} e^{-\lambda L} \int_{L-W}^{L} \lambda e^{\lambda u}\ du$$

$$+ \frac{c_0}{W} e^{-\lambda W} \int_{L-W}^{L} \lambda u e^{\lambda u}\ du$$

$$= c_0 - \frac{c_0}{W} \mu \left[1 - e^{-\lambda W}\right] - \frac{c_0 L}{W} \left[1 - e^{-\lambda W}\right]$$

$$+ \frac{c_0}{W} \left[L - \mu - Le^{-LW} + We^{-\lambda W} + \mu e^{-\lambda W}\right]$$

$$= c_0 - 2 \frac{c_0}{W} \mu F(W) + c_0\ e^{-\lambda W}\ .$$

Substituting in to the formula for $C(L)$

$$C(L) = c_0\ (1+M(L)) + c_0\left[\frac{\mu_W}{W} - F(W)\right] [1 + M\ (L - W)]$$

$$- c_0\ \lambda W + c_0 F(W) + c_0 - \frac{c_0}{W} \mu F(W) - \frac{c_0}{W} \mu F(W) + c_0\ e^{-\lambda W}$$

$$= c_0\left[1 + \lambda L + \frac{\mu_W}{W} + \frac{\mu_W}{W}\,\lambda(L-W) - \lambda W + 1 + e^{-\lambda W}\right]$$

$$+ c_0\; F(W)\left[\lambda W - \lambda L - 2\,\frac{\mu}{W}\right].$$

As expected, this equation is linear in L.

To get a feel for the meaning of this equation let $\lambda = 1$ (W.L.O.G.). This can be done by use of an appropriate time scale without affecting the results. C(L) now simplifies to

$$C(L) = c_0\left[1 + L + \frac{\mu_W}{W}\,[1+L-W] - W + 1 + e^{-W}\right]$$

$$+ c_0\left[1 - e^{-W}\right]\left[W - L - \frac{2}{W}\right]$$

$$= c_0\left(1 + e^{-W} - \frac{1}{W}\left[1-e^{-W}\right] + \frac{L}{W}\left[1-e^{-W}\right]\right) \qquad L \geq W\ .$$

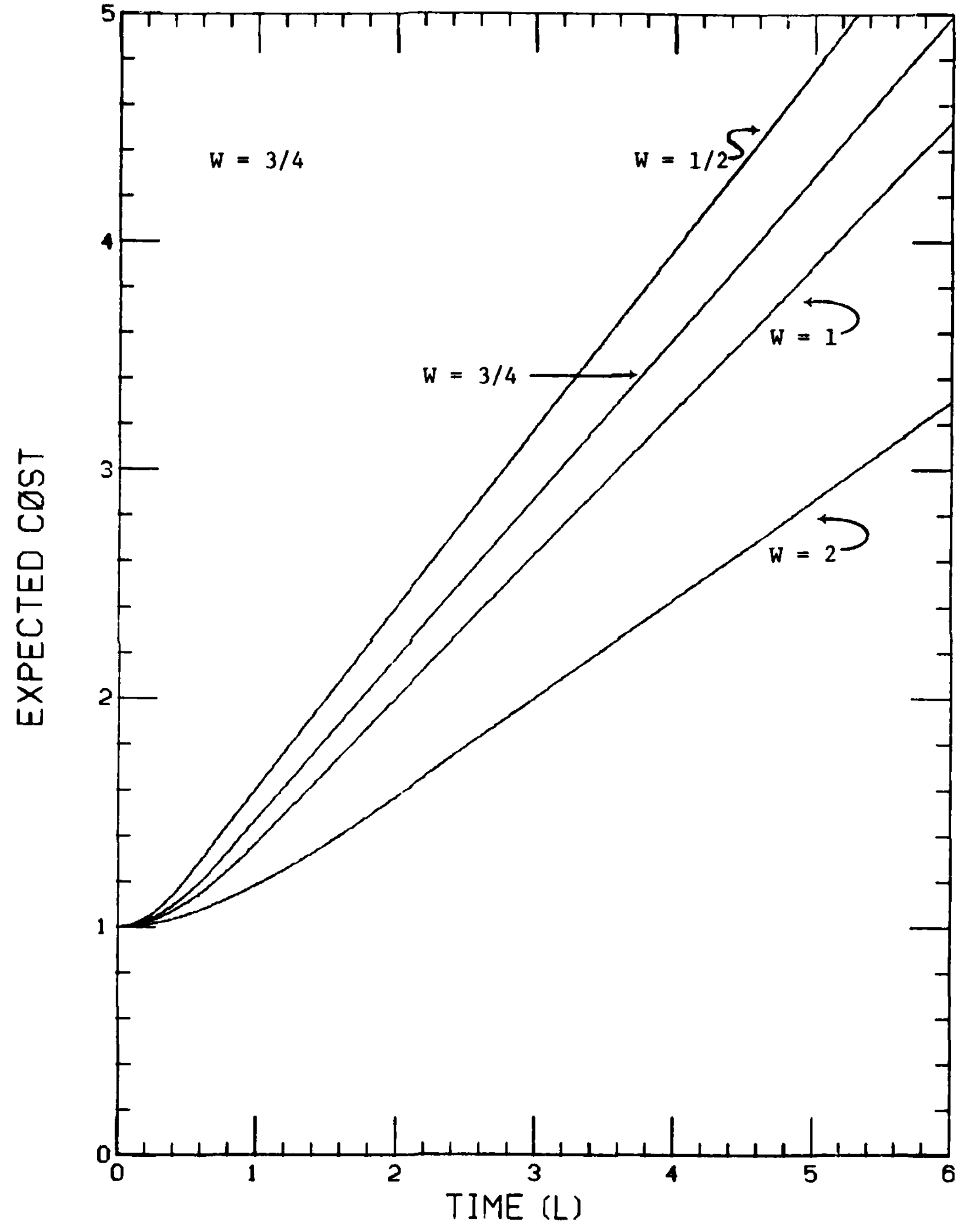
Recall that L is assumed to be greater than or equal to W for this formulation. If $L < W$ a similar analysis can be performed

$$C(L) = c_0 + \frac{c_0}{W}\,\mu_L + \int_0^L \frac{c_0}{W}\,\mu_{L-u}\,\lambda du$$

$$= c_0\left[1 + \frac{L}{W} - \frac{1}{W}\left(1-e^{-L}\right)\right] \qquad L < W\ .$$

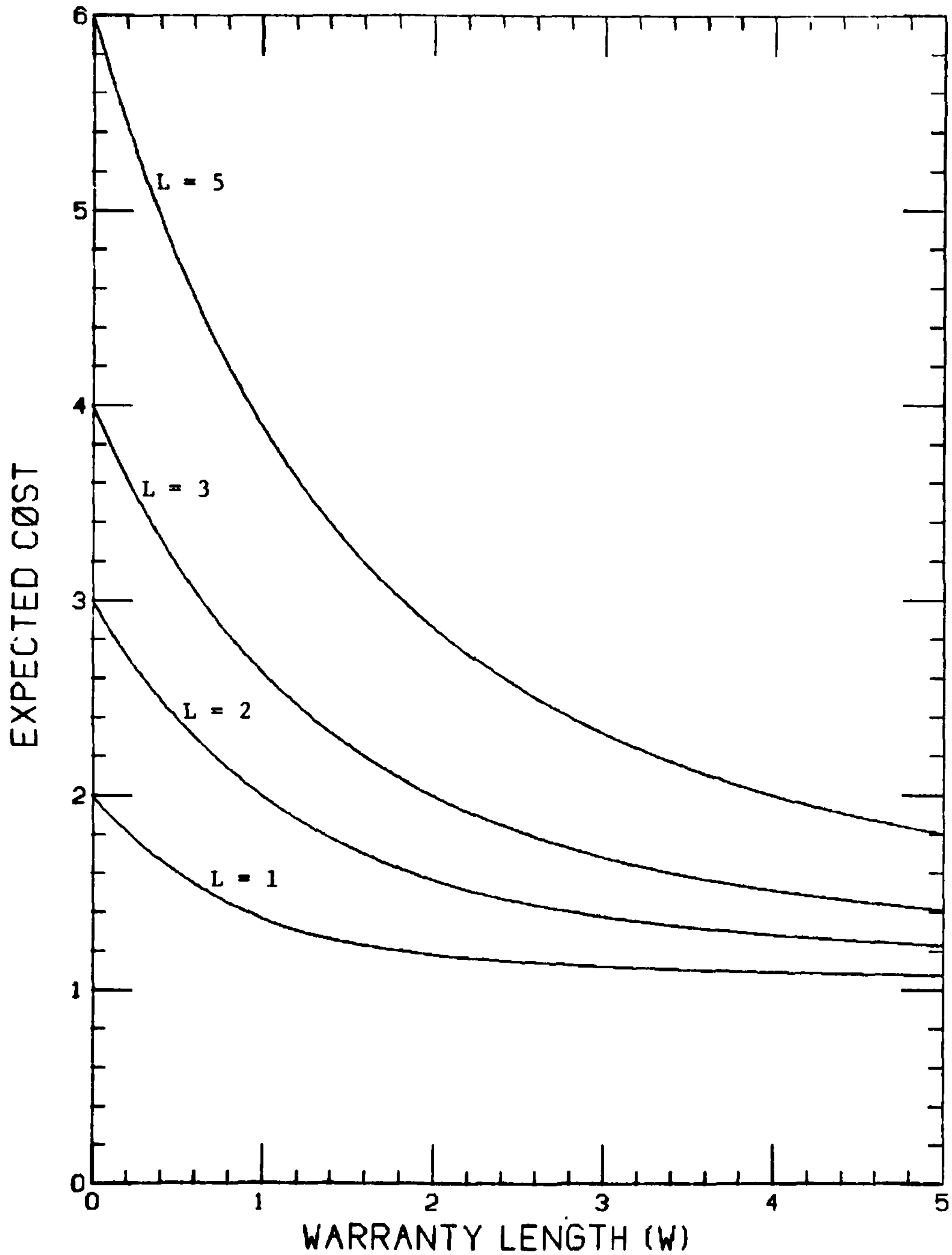
$C(L)$ is plotted vs. L for various values of W in Figure 3.3.1. Note that $C(L)$ drops off significantly as W increases. This is further verified by Figure 3.3.2, a plot of $C(L)$ vs. W for various values of L. In both graphs there is a discontinuity (in the second derivative) at $L = W$.

FIGURE 3.3.1



Expected Cost to the Consumer vs. Time for the Pro Rata Warranty under Various Warranty Length Assumptions (assumes exponential life lengths).

FIGURE 3.3.2



Expected Cost to the Consumer vs. Warranty Length for the Pro Rata Warranty under Various Time Assumptions (assumes exponential life lengths).

3.4. The Pro Rata Warranty with Delay

In the first generalization of the pro rata warranty policy the customer is not charged if the item fails before time s. After time s but before time W the customer is charged pro rata for the time after s. Figure 3.4 displays graphically the amount the customer must pay for a new item as a function of the life length of the previous item.

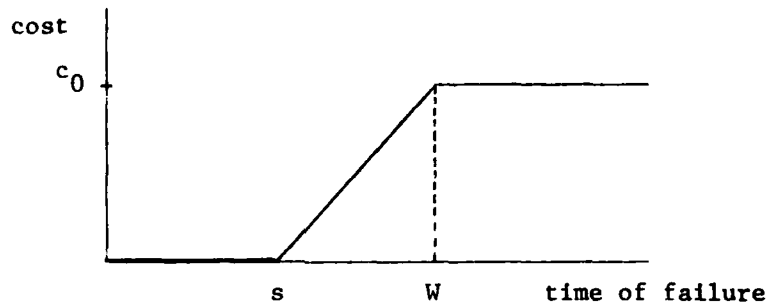


Figure 3.4. Replacement Cost: First Generalization

As with all policies discussed in Chapter 3 a new warranty is issued with each new item.

An important feature of this policy is that the consumer receives a free replacement of the original item whenever the customer's current item fails before operating for at least s units of time. In this sense the policy acts exactly like the renewing warranty policy analyzed in Section 3.2.

From the manufacturer's point of view, the costs up to time L including possible obligations at time L are found by the use of

Wald's Lemma. The renewal function is $M_G(L)$ as defined in Section 3.2 with s replacing the W due to the free replacement period now being $[0,s]$.

$$M_G(L) = M(L) - F(s) - F(s)\, M(L-s) - \int_{L-s}^{L} F(L-u)\, dM(u) \; .$$

The expected cost per cycle (recall that a cycle is defined as the time between payments by the consumer) is also as in Section 3.2

$$E[k_1] = \frac{k_0}{\bar{F}(s)}$$

and the expected cost to the manufacturer is

$$K(L) = E[k_1]\,(1 + M_G(L))$$

$$= \frac{k_0}{\bar{F}(s)} \left[1 + M(L) - F(s) - F(s)\, M(L-s) - \int_{L-s}^{L} F(L-u)\, dM(u) \right] .$$

$\tau(L)$, the useful life to the consumer is similarly found to be $\tau(1 + M_G(L))$, where $\tau = \frac{\mu}{\bar{F}(s)}$,

$$\tau(L) = \frac{\mu}{\bar{F}(s)} \left[1 + M(L) - F(s) - F(s)\, M(L-s) - \int_{L-s}^{L} F(L-u)\, dM(u) \right] .$$

From the consumer's point of view the appropriate cost function $Y(x)$ as shown in Figure 3.4 is,

$$Y(x) = \begin{cases} 0 & x \le s \\ c_0 \frac{x-s}{W-s} & s \le x \le W \\ c_0 & x \ge W \end{cases} .$$

$V(T)$, the expected cost of the first renewal up to time T, is

$$V(T) = \int_0^T E(Y(x) \mid x = u)\, dF(u)$$

$$= \begin{cases} 0 & T \le s \\ \int_s^T c_0 \frac{u-s}{W-s}\, dF(u) & s \le T \le W \\ \int_s^W c_0 \frac{u-s}{W-s}\, dF(u) + \int_W^T c_0\, dF(u) & T \ge W \end{cases}$$

The renewal equation as derived in Section 2.1 is once again

$$E(R(T)) = V(T) + \int_0^T E(R(T-u))\, dF(u) ,$$

where $R(T)$ is the total cost to the consumer up to time T not including the initial cost c_0. The solution is

$$E(R(T)) = V(T) + \int_0^T V(T-u)\, dM(u)$$

$$= \int_s^W c_0 \frac{u-s}{W-s}\, dF(u) + c_0 \int_W^T dF(u)$$

$$+ \int_0^{T-W} dM(u) \cdot \int_s^W c_0 \frac{u-s}{W-s}\, dF(u) + \int_0^{T-W} \int_W^{T-u} c_0\, dF(x)\, dM(u)$$

$$+ \int_{T-W}^{T-s} \int_s^{T-u} c_0 \frac{x-s}{W-s}\, dF(x)\, dM(u)$$

$$+ \int_{T-s}^{T} 0 \cdot dM(u)$$

$$= \frac{c_0}{W-s} [\mu_W - \mu_s] - \frac{c_0 s}{W-s} [F(W)-F(s)] + c_0[F(T)-F(W)]$$

$$+ M(T-W) \left[\frac{c_0}{W-s} [\mu_W - \mu_s] - \frac{c_0 s}{W-s} [F(W)-F(s)]\right]$$

$$+ c_0\, M(T) - c_0\, F(T) - c_0 \int_{T-W}^{T} F(T-u)\, dM(u) - F(W)\, M(T-W)$$

$$+ \int_{T-W}^{T-s} \frac{c_0}{W-s} [\mu_{T-u} - \mu_s]\, dM(u) - \frac{c_0 s}{W-s} \int_{T-W}^{T-s} [F(T-u)-F(s)]\, dM(u) .$$

Combining similar terms

$$E(R(T)) = c_0\, M(T) + c_0 \left(\frac{\mu_W}{W-s} - F(W)\right)(1+M(T-W))$$

$$+ \frac{c_0}{W-s} \int_{T-W}^{T-s} \mu_{T-u}\, dM(u) - c_0 \int_{T-W}^{T} F(T-u)\, dM(u)$$

$$+ \frac{c_0}{W-s} \left[- \mu_s - \mu_s\, M(T-W) - \int_{T-W}^{T-s} \mu_s\, dM(u)\right]$$

$$+ \frac{c_0 s}{W-s} \left[F(s)(1+M(T-s)) - F(W)(1+M(T-W)) - \int_{T-W}^{T-s} F(T-u)\, dM(u)\right] .$$

$C(L)$, the expected cost to the consumer up to time L, is

$$C(L) = c_0 + E(R(L)) ,$$

where $E(R(L))$ is as above. Unfortunately, this is quite a complicated expression. For large L this can be approximated by $\hat{c}L$ where $\hat{c} = \lim_{T\to\infty} \frac{c(T)}{T}$.

Lemma 3.4.1. For the delayed pro rate warranty policy

$$\hat{c} = \frac{\frac{c_0}{W-s} [\mu_W - \mu_s + sF(W) - sF(s)] + c_0(1-F(W))}{\mu} .$$

Proof: $V(T)$ is a bounded nondecreasing function so by Theorem 2.2, $c^* = \frac{V^*}{\mu}$ where $V^* = \lim_{t \to \infty} V(t)$. ☒

3.5. The Pro Rata Warranty with Rebate

The second generalization of the pro rata warranty also has a "free period" during which the consumer receives a free item. However, after the free period the consumer must pay the full pro rata cost. The term rebate applies because

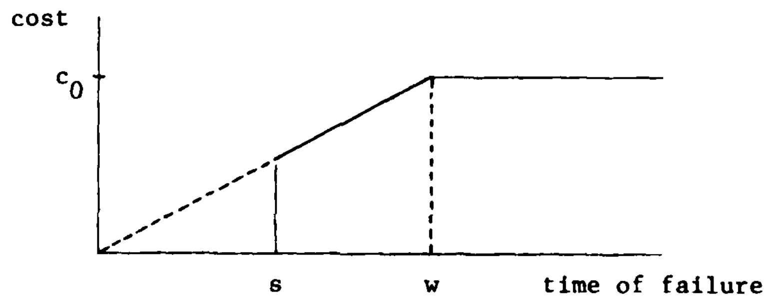


Figure 3.5

Replacement Cost: Second Generalization

this policy is effectively a standard pro rata policy with the customer receiving a rebate of the pro rata cost whenever the item fails before time s. Figure 3.5 displays graphically the amount the consumer must pay as a function of the life length of the previous item.

The expected cost to the manufacturer and process length are identical to those for the pro rata warranty with delay as discussed in Section 3.4. That is

$$K(L) = \frac{k_0}{\bar{\bar{F}}(s)} (1 + M_G(L))$$

and

$$\tau(L) = \frac{\mu}{\bar{\bar{F}}(s)} (1 + M_G(L))$$

where

$$M_G(L) = 1 + M(L) - F(s) - F(s)\, M(L-s) - \int_{L-s}^{L} F(L-u)\, dM(u) \ .$$

The expected cost to the consumer is found by noting that the cost function

$$Y(x) = \begin{cases} 0 & x \leq s \\ \frac{c_0}{W} x & s \leq x \leq W \\ c_0 & x \geq W \ . \end{cases}$$

Following the derivation in Section 2.1,

$$V(T) = \begin{cases} 0 & T \leq s \\ \frac{c_0}{W} [\mu_W - \mu_s] & s \leq T \leq W \\ \frac{c_0}{W} [\mu_W - \mu_s] + c_0 \int_W^T dF(x) & T \geq W \end{cases}$$

and the solution to the renewal equation is

$$E[R(T)] = V(T) + \int_0^T V(T-u)\, dM(u)$$

$$= \frac{c_0}{W}[\mu_W - \mu_s] + c_0 [F(T) - F(W)]$$

$$+ \int_0^{T-W} \frac{c_0}{W}[\mu_W - \mu_s]\, dM(u) + \int_0^{T-W} c_0[F(T-u) - F(W)]\, dM(u)$$

$$+ \int_{T-W}^{T-s} \frac{c_0}{W}[\mu_{T-u} - \mu_s]\, dM(u)$$

$$= \frac{c_0}{W}[\mu_W - \mu_s] + c_0 [F(T) - F(W)]$$

$$+ \frac{c_0}{W}[\mu_W - \mu_s]\, M(T-W) - c_0\, F(W)\, M(T-W)$$

$$+ c_0\, M(T) - c_0\, F(T) - c_0 \int_{T-W}^{T} F(T-u)\, dM(u)$$

$$- \frac{c_0}{W}\mu_s[M(T-s) - M(T-W)] + \frac{c_0}{W}\int_{T-W}^{T-s} \mu_{T-u}\, dM(u) .$$

Adding the initial term and simplifying somewhat,

$$C(L) = c_0\, (1 + M(L)) + c_0 \left(\frac{\mu_W}{W} - F(W)\right)(1 + M(L-W))$$

$$+ \frac{c_0}{W}\int_{L-W}^{L-s} \mu_{L-u}\, dM(u) - c_0 \int_{L-W}^{L} F(L-u)\, dM(u)$$

$$- \frac{c_0}{W} \mu_s (1 + M(L-s)) .$$

If $s = 0$ the first four terms are exactly $C(L)$ for the standard pro rata warranty and the last term is zero since $\mu_0 = 0$.

The difference in consumer cost between the standard pro rata warranty and the pro rata warranty with rebate is now easily seen to be

$$\frac{c_0}{W} \int_{L-s}^{L} \mu_{L-u} \, dM(u) + \frac{c_0}{W} \mu_s (1 + M(L-s)) .$$

Thus, if the consumer intends to stay in the system for L units of time, this is the additional cost the consumer should be willing to pay to add the rebate clause to his warranty (and to all future warranties).

CHAPTER 4

NONRENEWING WARRANTY POLICIES

Certain warranty policies do not start over or renew themselves whenever a new item is issued. These policies include the standard warranty policy and the generalization of it discussed in Section 1.2. For these policies, the times of issuance of new items do not form a regenerative process.

If, however, a new warranty is offered whenever there is a cash transaction, that is, whenever the consumer must pay for a replacement, then the times of payment generate a regenerative process. This new process will have a different renewal function associated with it along with different costs. Let this new function be $M_G(t)$ with $M_G(0) = 0$ by convention.

In each of the policies in this chapter the consumer either pays the full price c_0 to replace the item or else receives the new item free of charge. Since the times of payment represent renewal times the total expected cost to the consumer is

$$C(L) = c_0 \; (1 + M_G(L)) \; .$$

Likewise, the total expected cost to the manufacturer and total expected useful life length are found to be

$$K(L) = E(k_1) \; (1 + M_G \; (L))$$

and

$$\tau(L) = \tau(0)\ (1 + M_G\ (L))\ .$$

Since $M_G(L)$ cannot be explicitly calculated except in special circumstances, (i.e., $F(x)$ distributed exponentially) and even then it is extremely complex, the limiting approximation $M_G(L) \simeq \frac{L}{\tau}$ will be used when comparing the manufacturer's profit per customer, $P(L) = C(L) - K(L) = [c_0 - E(k_1)]\ [1 + M_G(L)]$.

The other comparisons are not affected due to the $\tau(L)$ term in the denominator which cancels the $1 + M_G(L)$ term.

If additional information is known about G, the distribution function governing the cycle lengths, a better estimate of $M_G(L)$ can be made. (A different proof to Theorem 4.1 can be found in Feller, Vol. II [11].)

Theorem 4.1. If G is a nonarithmetic distribution with finite variance, $\sigma_G^2 = E(Z - \tau)^2$, then

$$\lim_{L\to\infty} \left\{ M_G(L) - \frac{L}{\tau} \right\} = \frac{\sigma_G^2 - \tau^2}{2\tau^2}\ .$$

Proof: Let

$$h(L) = \tau M_G(L) + \tau - L\ .$$

Then

$$\int_0^L h(L-u)\ dG(u) = \tau \int_0^L M_G(L-u)\ dG(u) + \tau G(L) - \int_0^L (L-u)\ dG(u)$$

$$= \tau M_G(L) - \tau G(L) + \tau G(L) - \int_0^\infty (L-u)\ dG(u)$$

$$+ \int_L^\infty (L-u)\ dG(u)$$

$$= \tau M_G(L) + \tau - L + \int_L^\infty (L-u)\ dG(u)\ .$$

Rearranging terms, one notices a renewal equation

$$h(L) = \int_L^\infty (u-L)\ dG(u) + \int_0^L h(L-u)\ dG(u)\ .$$

The first part is monotone so by the Basic Renewal Theorem:

$$\lim_{L\to\infty} h(L) = \frac{1}{\tau} \int_0^\infty \int_L^\infty (u-L)\ dG(u)\ dL$$

$$= \frac{1}{\tau} \int_0^\infty \int_0^u (u-L)\ dL\ dG(u)$$

$$= \frac{1}{\tau} \int_0^\infty \frac{u^2}{2}\ dG(u)$$

$$= \frac{1}{2\tau}\left[\sigma_G^2 + \tau^2\right] .$$

Recalling

$$M_G(L) - \frac{L}{\tau} = \frac{h(L)}{\tau} - 1 ,$$

$$\lim_{L\to\infty}\left\{M_G(L) - \frac{L}{\tau}\right\} = \frac{\sigma_G^2+\tau^2}{2\tau^2} - 1$$

$$= \frac{\sigma_G^2-\tau^2}{2\tau^2} . \qquad \boxtimes$$

Thus, if σ_G^2 is known (and typically it is not) a better estimate of $M_G(L)$ is

$$M_G(L) \approx \frac{L}{\tau} + \frac{\sigma_G^2-\tau^2}{2\tau^2} .$$

4.1 The Standard Warranty Policy

The standard warranty policy is the fixed time warranty. It is typically offered with such items as automobiles, stereos, televisions, washing machines and other complex, expensive devices. In this policy whenever the purchased item fails before time W, the item is replaced free of charge. The warranty, however, is not renewed. If the next item fails and the sum of the failure times of both the first and second item is less than W (i.e., the warranty has not expired yet), then another item will be issued once again free of charge. This continues until the total time is greater than W. At that point no more free items are issued and the consumer must pay the full price for a new item.

The distribution of the first time of failure after time W can be calculated via a renewal argument. Let $\gamma(W)$ be the remaining life of the current item at time W,

$$\gamma(W) = s_{N(W)+1} - W$$

and let

$$p_t(W) = \Pr\{\gamma(W) \geq t\} .$$

If x_1 is the time of the first failure then

$$p_t(W) = \begin{cases} 1 & x_1 \geq W + t \\ 0 & W \leq x_1 < W + t \\ p_t(W-x_1) & x_1 < W \end{cases} .$$

Conditioning upon x_1,

$$p_t(W) = \int_{W+t}^{\infty} t \cdot dF(x) + \int_{W}^{W+t} 0 \cdot dF(x) + \int_{0}^{W} p_t(W-x)\, dF(x)$$

$$= \bar{F}(W + t) + \int_{0}^{W} p_t(W-x)\, dF(x) .$$

$p_t(W)$ can now be solved for by use of the renewal theorem

$$p_t(W) = \bar{F}(W+t) + \int_0^W \bar{F}(W+t-x)\, dM(x) \ .$$

$G(t)$, the distribution function governing the cycle lengths, is found by noting

$$G(t) = \begin{cases} 0 & t < W \\ 1 - p_t(W) & t \geq W \ . \end{cases}$$

Using this one can theoretically find σ_G^2,

$$\sigma_G^2 = \int_0^\infty t^2\, dG(t) - \tau^2$$

where τ is derived later in this section. σ_G^2 is a necessary value to determine the bias in estimating $M_G(L)$ by $\frac{L}{\tau}$ (see Section 4.1).

Frequently $p_t(W)$ is difficult to calculate. In these cases knowledge of the distribution function of the items $F(t)$ can sometimes be used to get bounds on $p_t(W)$. The following theorem is found in Barlow and Proschan [2]. The proof is short and of interest.

Definition: New Better (Worse) Than Used. A distribution F is said to be new better than used, NBU, if

$$\bar{F}(x + y) \leq \bar{F}(x)\, \bar{F}(y) \qquad x \geq 0,\ y \geq 0 \ .$$

Likewise F is said to be new worse than used, NWU, if

$$\bar{F}(x + y) \geq \bar{F}(x)\,\bar{F}(y) \qquad x \geq 0,\ y \geq 0 .$$

<u>Theorem 4.1.1</u>. If F is NBU, then

$$p_t(W) \leq \bar{F}(t) .$$

<u>Proof</u>: If F is NBU, then

$$\begin{aligned} p_t(W) &= \bar{F}(W + t) + \int_0^W \bar{F}(w + t - x)\, dM(x) \\ &\leq \bar{F}(W)\,\bar{F}(t) + \bar{F}(t) \int_0^W \bar{F}(W - x)\, dM(x) \\ &= \bar{F}(t) \left[\bar{F}(W) + \int_0^W \bar{F}(W - x)\, dM(x)\right] \\ &= \bar{F}(t)\, [1 - M(W) + M(W)] \\ &= \bar{F}(t) . \end{aligned}$$

⊠

If F is NWU, the same relationship (and proof) holds with the inequalities reversed.

The total cost to the consumer over one period or cycle is c_0 (since the consumer only pays once, at the beginning of the cycle). The expected length of the cycle, τ, is the expected time of the first failure after time W, which is found by Wald's Lemma, $\tau = \mu(1 + M(W))$.

The expected cost to the manufacturer of a cycle is the cost of an item times the expected number of items issued during the cycle or

$$k_0(1 + M(W)) .$$

$M_G(L)$, the expected number of cycles up to time L is difficult to calculate explicitly. However, it is known that $\frac{M_G(L)}{L}$ converges to $\frac{1}{\tau}$ and, hence, for large L, $M_G(L)$ can be approximated by $\frac{L}{\mu(1+M(W))}$.

The expected cost to the consumer per unit of useful life is

$$\frac{C(L)}{\tau(L)} \approx \frac{c_0[1+M_G(L)]}{\tau[1+M_G(L)]} \approx \frac{c_0}{\tau} \approx \frac{c_0}{\mu(1+M(W))} .$$

The expected profit per consumer can be approximated by

$$P(L) \simeq L[c_0 - k_0 - k_0 M(W)]/\mu(1 + M(W))$$

and the expected profit per customer per unit time is

$$\frac{P(L)}{\tau(L)} \simeq \frac{c_0}{\mu(1+M(W))} - \frac{k_0}{\mu} .$$

Comparison 4.1. Comparing this policy with the no warranty policy, the price c_0^* such that the consumer is indifferent between the two policies is found by equating the expected costs per unit of useful life

$$\frac{c_0^*}{\mu[1+M(W)]} = \frac{c_0}{\mu} \qquad \text{or,}$$

$$c_0^* = c_0[1 + M(W)] \ .$$

The approximation to the price the manufacturer should charge to break even is

$$\frac{\hat{c}_0 L}{\mu(1+M(W))} - \frac{k_0 L}{\mu} \approx (c_0 - k_0)\ (1 + M(L))$$

or

$$\hat{c}_0 \approx \frac{(c_0-k_0)(1+M(L))\mu(1+M(W))}{L} + \frac{k_0}{\mu}\ (1 + M(W)) \ .$$

As $L \to \infty$

$$\hat{c}_0 \to (c_0 - k_0)(1 + M(W)) + k_0\ (1 + M(W)) = c_0\ (1 + M(W))$$

which is the same as c_0^* .

<u>Example 4.1</u>. If $\{x_i\}$ is distributed exponentially with parameter λ, then

$$\tau = \mu(1 + \lambda W) = \mu + W,$$

$$\frac{C(L)}{\tau(L)} = \frac{c_0}{\mu+W} \qquad \text{and}$$

$$P(L) \approx \left[\frac{c_0}{\mu+W} - \frac{k_0}{\mu}\right] L \; .$$

Other interesting results can be derived in this specialized case. The memoryless property of the exponential distribution implies that at time W, no matter how many items have been replaced thus far, the distribution of the current item from now on is exponential with parameter λ. Thus, a formula for $G(t)$, the distribution function for the cycle times, can be found.

$$G(t) = \begin{cases} 1 - e^{-\lambda(t-W)} & t \geq W \\ 0 & t < W \; . \end{cases}$$

Likewise, $G^{(n)}(t)$, the convolution of $G(t)$ with itself n times, is given by

$$G^{(n)}(t) = \begin{cases} \sum\limits_{k=n}^{\infty} \dfrac{e^{-\lambda(t-nW)}[\lambda(t-nW)]^k}{k!} & t \geq nW \\ 0 & t < nW \; . \end{cases}$$

$M_G(t)$, the expected number of cycles up to time t is thus found by the finite sum

$$M_G(t) = \sum_{n=1}^{\infty} G^{(n)}(t) = \sum_{1 \le n \le t/W} G^{(n)}(t) \; .$$

Making both sums finite,

$$M_G(t) = \sum_{1 \le n \le t/W} \left[1 - \sum_{k=0}^{n-1} \frac{e^{-\lambda(t-nW)}(\lambda(t-nW))^k}{k!} \right] .$$

In the introduction to this chapter it was shown that

$$\lim_{L \to \infty} \left[M_G(L) - \frac{L}{\tau} \right] = \frac{\sigma^2 - \tau^2}{2\tau^2}$$

where

$$\sigma^2 = E[Z_1 - \tau]^2, \qquad (Z_1 \text{ is the cycle length}).$$

In this example, due to the properties of the exponential distribution, σ^2 is equal to λ^2. A good approximation to $M_G(L)$ is thus

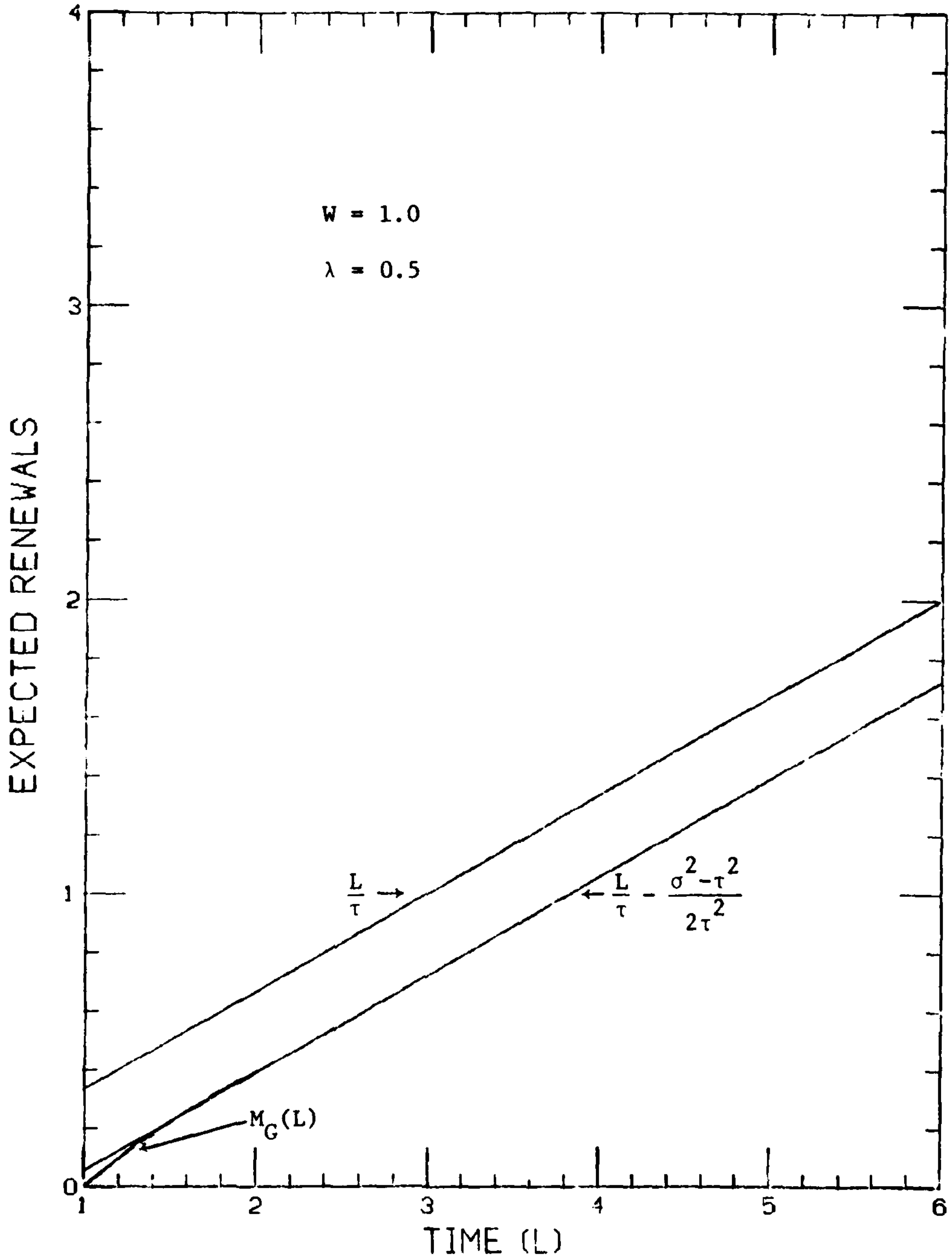
$$M_G(L) \approx \frac{L}{\mu+W} - \frac{\lambda^2 - (\mu+W)^2}{2(\mu+W)^2} \; .$$

To check the accuracy of this approximation,

$$M_G(L), \quad \frac{L}{\tau} \quad \text{and} \quad \frac{L}{\tau} - \frac{\sigma^2 - \tau^2}{2\tau^2}$$

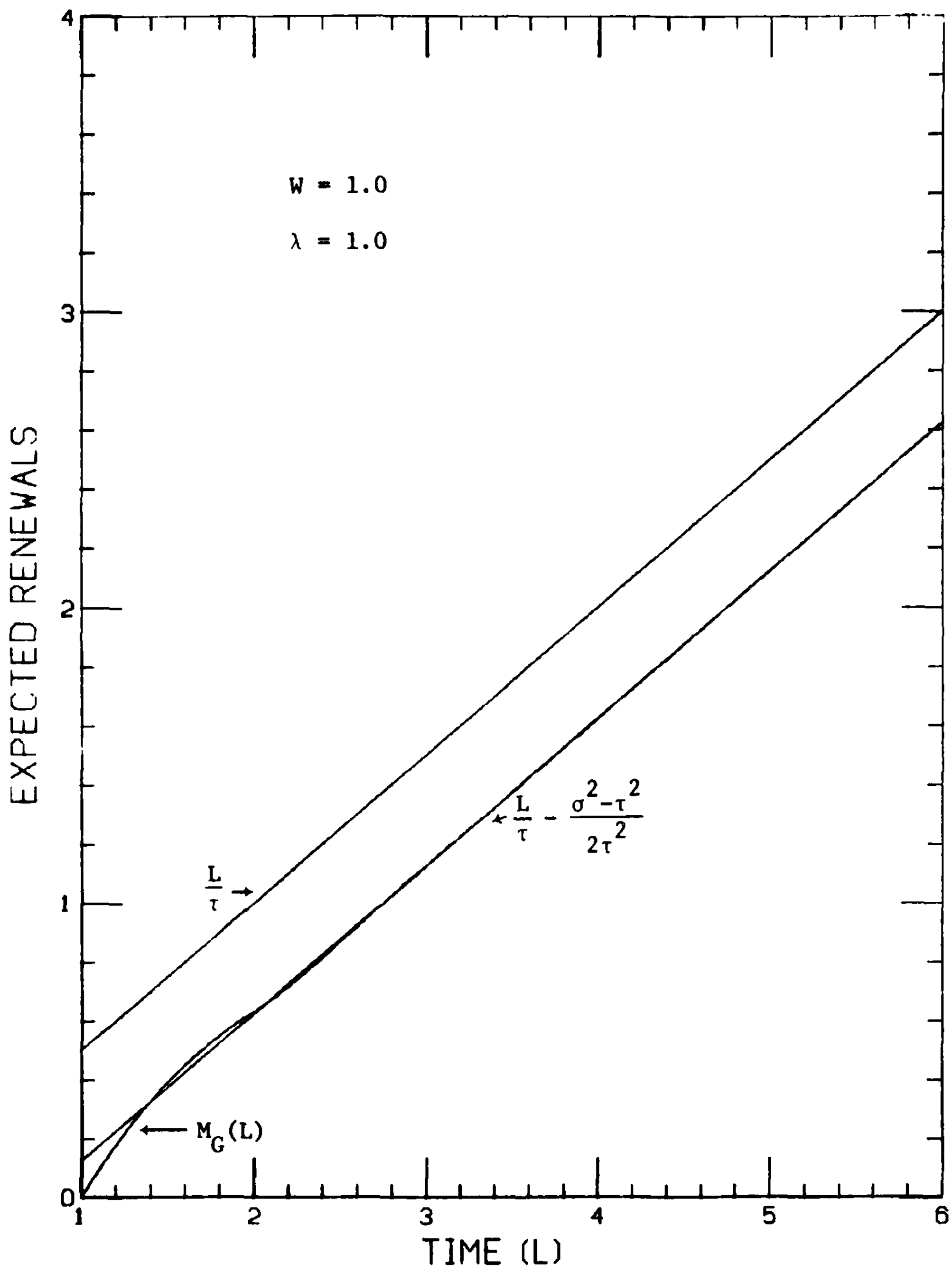
are plotted in Figures 4.1.1 - 4.1.6.

FIGURE 4.1.1



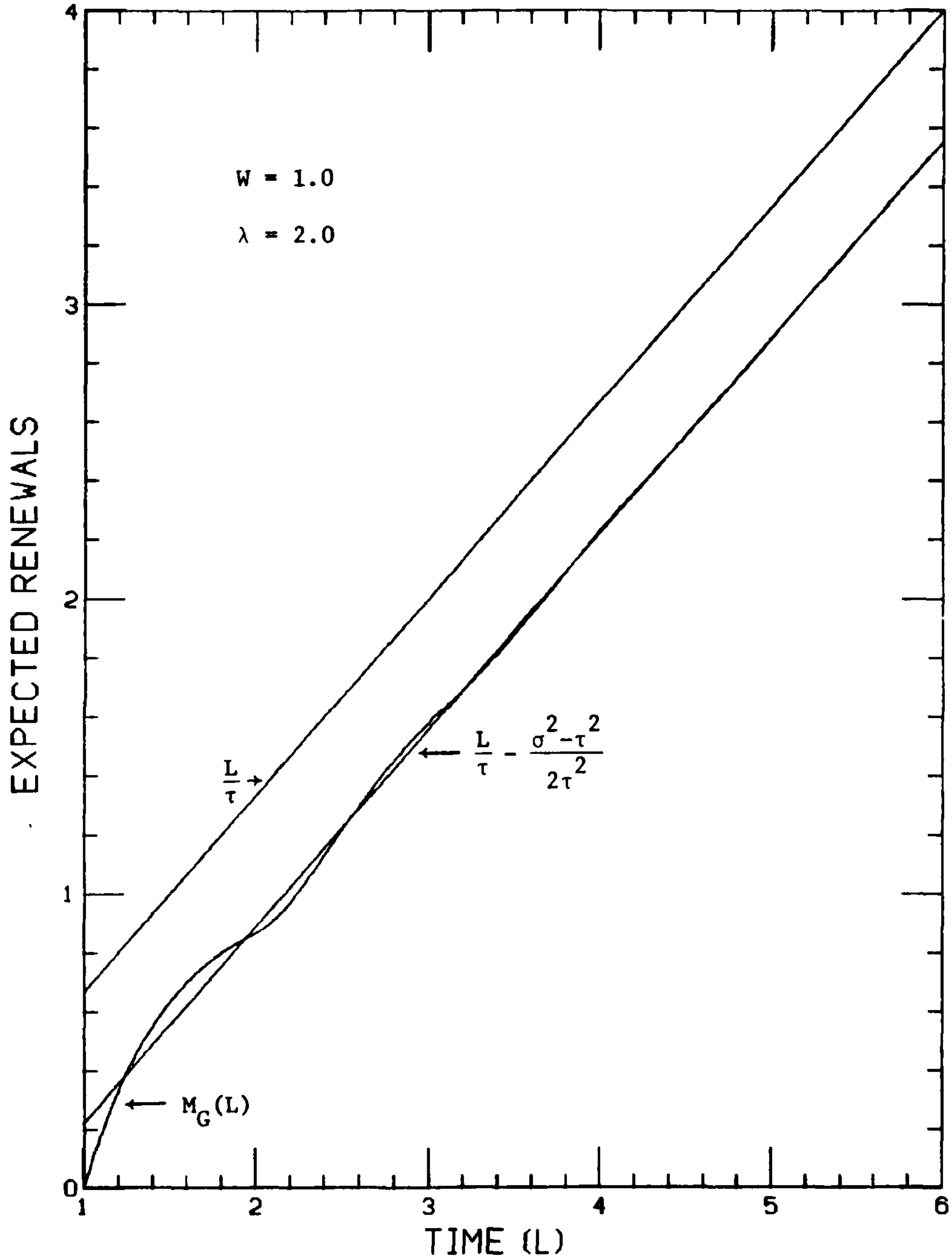
Expected Number of Renewals vs. Time for the Standard Warranty (assumes exponential life lengths).

FIGURE 4.1.2



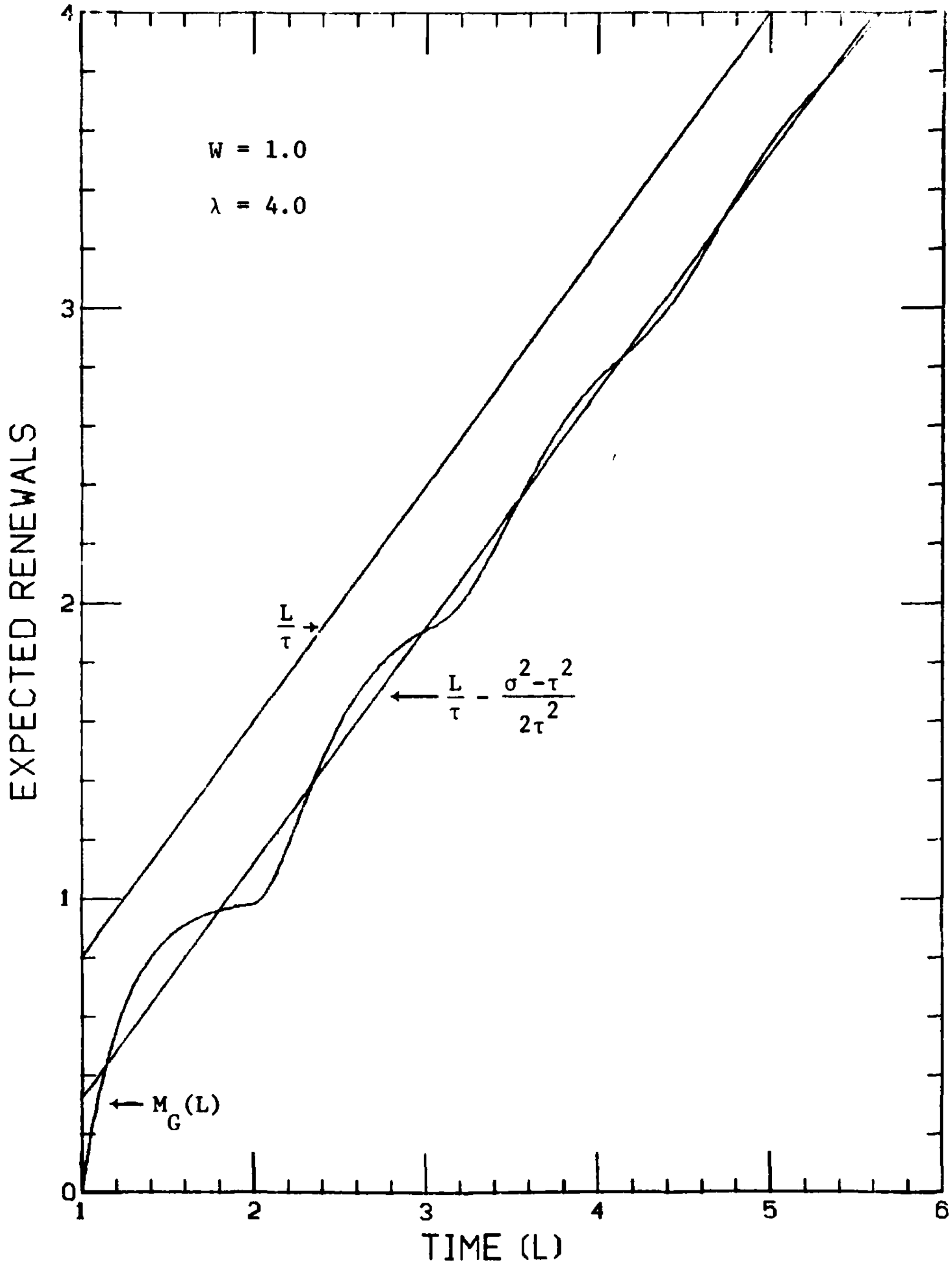
Expected Number of Renewals vs. Time for the Standard Warranty Policy (assumes exponential life lengths).

FIGURE 4.1.3



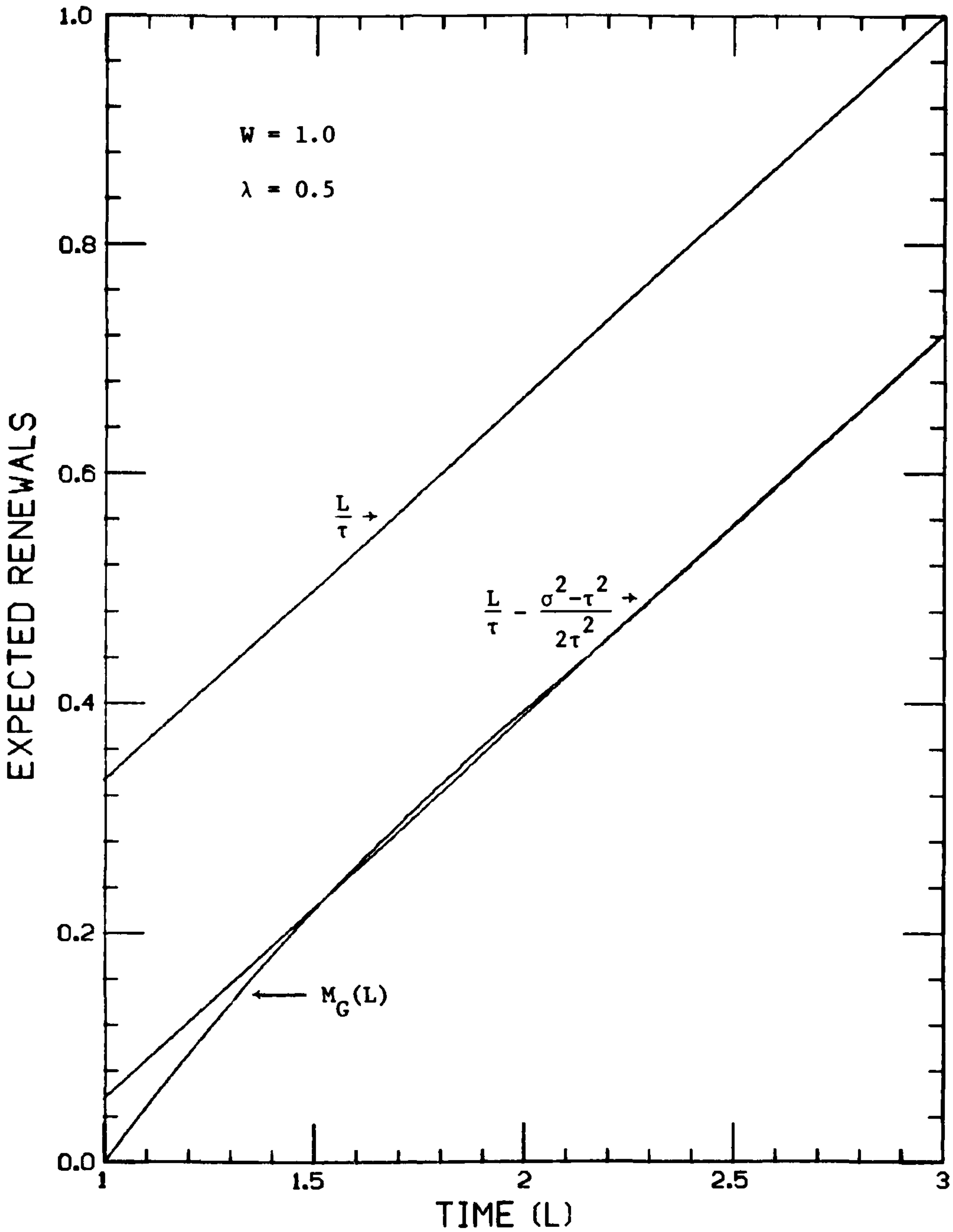
Expected Number of Renewals vs. Time for the Standard Warranty (assumes exponential life lengths).

FIGURE 4.1.4



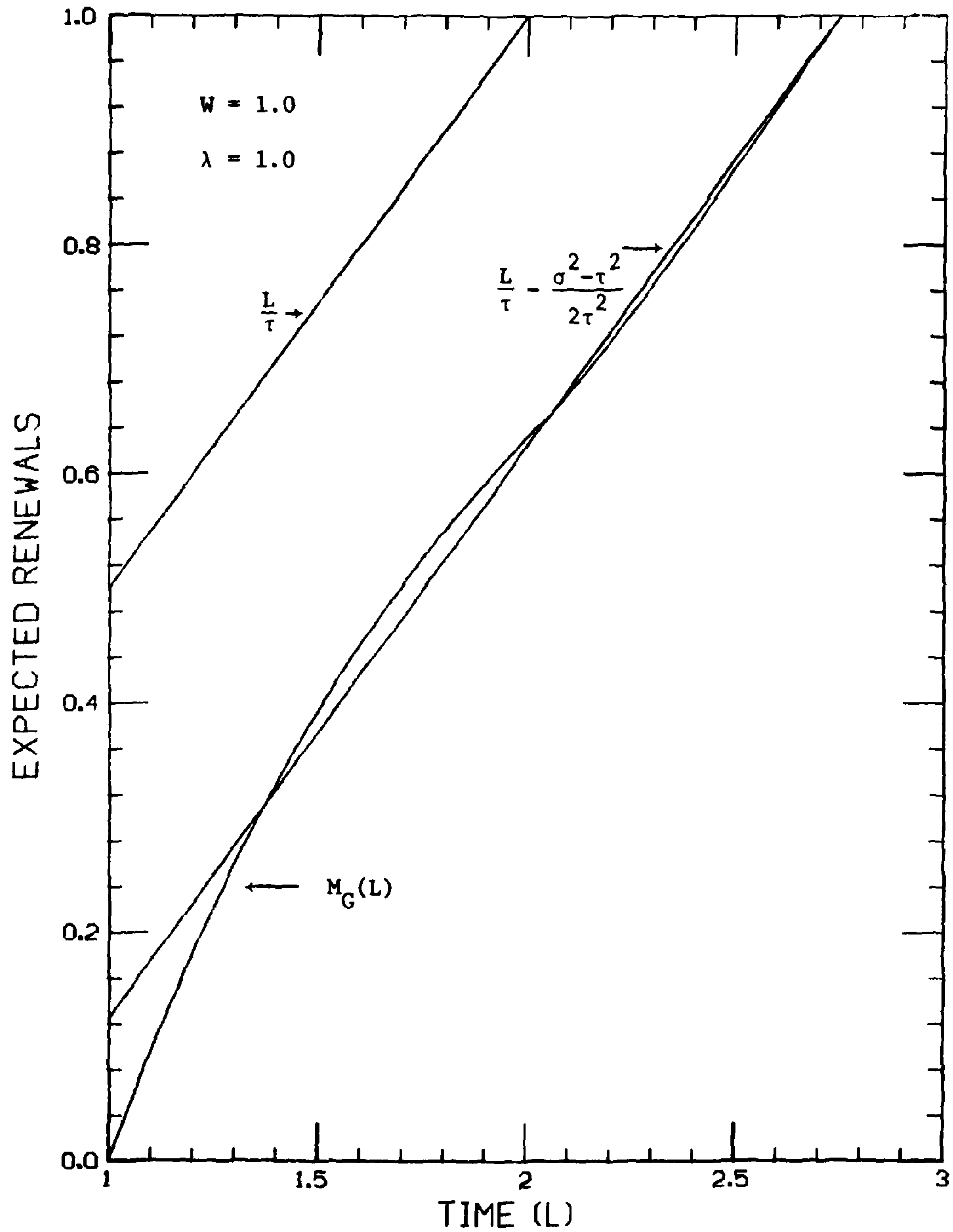
Expected Number of Renewals vs. Time for the Standard Warranty Policy (assumes exponential life lengths).

FIGURE 4.1.5



Expected Number of Renewals vs. Tim[illegible] or the Standard Warranty (assumes exponential life lengths).

FIGURE 4.1.6



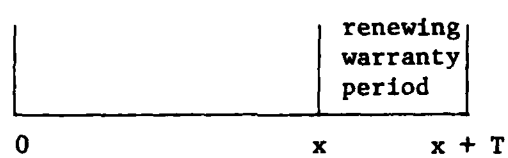
Expected Number of Renewals vs. Time for the Standard Warranty (assumes exponential life lengths).

4.2. The (T,W) Warranty Policy

The (T,W) warranty policy is a generalization of both the standard warranty policy and the renewing warranty policy. It is currently being used by such major electronics firms as Texas Instruments. It provides greater consumer protection than the standard warranty but less than the renewing warranty.

The initial item or product is issued with a full warranty of length W. If the item fails at time $t_1 < W$, then the item is replaced by the manufacturer, free of charge, and a new warranty of length $\max(T, W - t_1)$ is issued with the item.

As in the other nonpro rata policies discussed in this paper, the total cost to the consumer over one cycle is exactly c_0. The other values of interest, the expected cycle length, τ, and the expected cost to the manufacturer of a cycle are found by renewal arguments. To simplify the necessary equations a slightly different set of notation will be used.



Let the total warranty length be $x + T$ where the interval $(x, x + T)$ represents the period of time during which a renewing warranty policy of length T is in effect. It is known from Chapter 3 that if an item fails during this interval the additional expected cost to the manufacturer is $\frac{k_0}{\bar{F}(T)}$ and the additional expected cycle length is $\frac{\mu}{\bar{\bar{F}}(T)}$. If $A(x)$ is the total expected cycle length of this policy and t_1 is the time of the first failure, then by conditioning upon t_1

$$A(x) = \int_0^x t_1 + A(x-t_1)\, dF(t_1) + \int_x^{x+T} t_1 + \frac{\mu}{\bar{\bar{F}}(T)}\, dF(t_1) + \int_{x+T}^{\infty} t_1\, dF(t_1)$$

$$= \int_0^{\infty} t_1\, dF(t_1) + \frac{\mu}{\bar{\bar{F}}(T)} \int_x^{x+T} dF(t_1) + \int_0^x A(x - t_1)\, dF(t_1)$$

$$= \mu + \frac{\mu}{\bar{\bar{F}}(T)} (F(x + T) - F(x)) + \int_0^x A(x - t)\, dF(t) .$$

This is a standard renewal equation with T fixed and x as the variable. It is solved by

$$A(x) = \mu + \frac{\mu}{\bar{\bar{F}}(T)} (F(x + T) - F(x)) + \mu M(x)$$

$$+ \frac{\mu}{\bar{\bar{F}}(T)} \int_0^x F(x + T - u) - F(x - u)\, dM(u) .$$

τ, the expected cycle length, is clearly seen to be $A(W - T)$.

$$\tau = \mu(1 + M(W - T)) + \frac{\mu}{\bar{\bar{F}}(T)} (F(W) - F(W - T))$$

$$+ \frac{\mu}{\bar{\bar{F}}(T)} \int_0^{W-T} F(W-u) - F(W - T - u)\quad dM(u) .$$

The expected cost to the manufacturer of a cycle, $E(k_1)$, is likewise found from the renewal equation

$$B(x) = \int_0^x k_0 + B(x - t)\, dF(t) + \int_x^{x+T} \frac{k_0}{\bar{F}(T)}\, dF(t) + \int_{x+T}^{\infty} 0 \cdot dF(t)$$

$$= k_0\, F(x) + \frac{k_0}{\bar{F}(T)}\, (F(x + T) - F(x)) + \int_0^x B(x - t)\, dF(t)$$

where $B(x)$ represents the total expected cost (not counting the cost at time zero) to the manufacturer of a cycle. Solving for $B(x)$,

$$B(x) = k_0\, F(x) + \frac{k_0}{\bar{F}(T)}\, (F(x + T) - F(x)) + k_0 \int_0^x F(x - u)\, dM(u)$$

$$+ \frac{k_0}{\bar{F}(T)} \int_0^x F(x + T - u) - F(x - u)\, dM(u)$$

$$= k_0\, M(x) + \frac{k_0}{\bar{F}(T)}\, (F(x + T) - F(x))$$

$$+ \frac{k_0}{\bar{F}(T)} \int_0^x F(x + T - u) - F(x - u)\, dM(u)\ .$$

The expected cost of a cycle is

$$E(k_1) = k_0 + B(W - T)\ .$$

It is not currently known how to calculate $M_G(L)$, the expected number of cycles up to time L. Nor is it known how to find σ_G^2 the variance of the distribution function governing $M_G(L)$. However, the approximations of the last section are still valid,

$$\frac{C(L)}{\tau(L)} = \frac{c_0}{\tau} ,$$

and

$$\frac{P(L)}{\tau(L)} \approx \frac{c_0}{\tau} - \frac{k_0+B(W-T)}{\tau} .$$

Example 4.2. If the life lengths of the individual items are distributed exponentially with parameter λ, then

$$A(x) = \mu(1 + \lambda x) + \mu e^{\lambda T}(e^{-\lambda x} - e^{-\lambda(x+T)})$$

$$+ \mu e^{\lambda T} \int_0^x e^{-\lambda(x-u)} - e^{-\lambda(x+T-u)} \lambda du$$

$$= \mu + x + \mu e^{-\lambda(x-T)} - \mu e^{-\lambda x} + \mu e^{\lambda T} e^{-\lambda x} \int_0^x \lambda e^{\lambda u} du$$

$$- \mu e^{-\lambda x} \int_0^x \lambda e^{\lambda u} du$$

$$= \mu + x + \mu e^{-\lambda(x-T)} - \mu e^{-\lambda x} + \mu e^{\lambda T} - \mu e^{-\lambda(x-T)} - \mu + \mu e^{-\lambda x}$$

$$= x + \mu e^{\lambda T} .$$

Thus,

$$\tau = A(W - T) = W - T + \mu e^{\lambda T} .$$

Likewise,

$$B(x) = k_0 \lambda x + k_0 e^{\lambda T} (e^{-\lambda x} - \lambda e^{-\lambda(x+T)})$$

$$+ k_0 e^{\lambda T} \int_0^x e^{-\lambda(x-u)} - e^{-\lambda(x+T-u)} \lambda du$$

$$= k_0 (\lambda x + e^{\lambda T} - 1)$$

and

$$E(k_1) = k_0 (\lambda(W - T) + e^{\lambda T}) .$$

The expected profit per item sold is $c_0 - E(k_1)$ $= c_0 - k_0 (\lambda(W - T) + e^{\lambda T})$ and the expected profit per consumer per unit time is approximately

$$\frac{P(L)}{\tau(L)} \simeq \frac{c_0-E(k_1)}{\tau} = \frac{c_0-k_0\lambda(W-T)-k_0 e^{\lambda T}}{W-T+\mu e^{\lambda T}} .$$

For small λT the exponential term can be approximated by

$$e^{\lambda T} \simeq 1 + \lambda T + \frac{(\lambda T)^2}{2} ,$$

in which case

$$E(k_1) \simeq k_0 \left(\lambda w - \lambda T + 1 + \lambda T + \frac{(\lambda T)^2}{2} \right)$$

$$= k_0 \left(1 + \lambda W + \frac{(\lambda T)^2}{2} \right).$$

Recall from Section 4.1 that for a fixed warranty of length W, the total expected cost per cycle to the manufacturer was $k_0(1 + \lambda w)$. Thus, if T is small relative to the expected life length of an item ($\lambda T = \frac{T}{\mu}$ is small) the extra cost per item sold can be approximated by

$$\frac{k_0 T^2}{2\mu^2} .$$

This approximation is valid in many instances even when the life lengths are not strictly distributed exponentially. For instance, if the failure distribution is approximately exponential during the period $[0, W]$ with parameter $\hat{\lambda}$ then the approximation is still valid independent of the distribution after time W. In this case,

$\hat{\mu} = (\hat{\lambda})^{-1}$ will frequently be greater than or equal to μ and $\frac{k_0 T^2}{2\mu^2}$ will provide an upper bound on the expected additional cost per unit sold.

CHAPTER 5

THE OPTION OF REPAIRING

When an item under warranty fails, the manufacturer is often faced with the decision of whether or not to repair the item in lieu of replacing it. In this chapter it will be assumed that the manufacturer has decided to replace whenever more than a certain amount of time, s, remains in the warranty and repair whenever at most s is left in the warranty. This policy is depicted for the standard warranty policy in Figure 5.0. The ① and ② denote new and "used" items respectively.

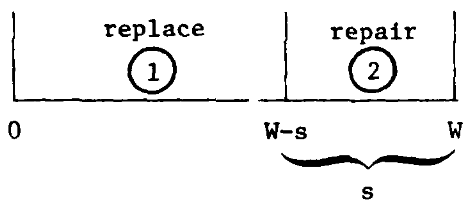


Figure 5.0
A Replace-Repair Policy for the Standard Warranty

In Chapter 6 conditions will be derived under which this type of policy is optimal and the question of repairing vs. replacing will be discussed in detail.

It will be assumed throughout that each time an item is repaired a fixed cost of c_2 is incurred by the manufacturer and the repaired item has a life distribution $F_2(t)$. For notational convenience, c_1

and $F_1(t)$ will represent the manufacturer's cost and life distribution of the replaced (or new) item, respectively.

5.1. The Renewing Warranty Policy

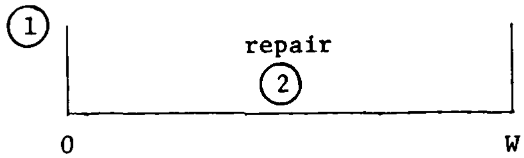


Figure 5.1

The Repair-Replace Policy for a Renewing Warranty

Recall that in the renewing warranty policy whenever a new item is issued a new warranty of length W is also issued. Since the replace-only option has previously been examined (see Section 3.2), the only repair-replace policy which needs to be considered is the policy with $s = W$.

The quantities of interest are still τ and $E(k_1)$. To find these it is necessary to condition upon the first time of failure, t, and use the formulae derived in Section 3.2.

$$\tau = \int_0^W t + \frac{\mu_2}{\bar{F}_2(W)} \, dF_1(t) + \int_W^\infty t dF_1(t)$$

$$= \mu_1 + \frac{\mu_2}{\bar{F}_2(W)} \, F_1(W) \; .$$

$$E[k_1] = c_1 + \int_0^W \frac{c_2}{\bar{F}_2(W)}\, dF_1(t) + \int_W^\infty 0 \cdot dF_1(t)$$

$$= c_1 + \frac{c_2 F_1(W)}{\bar{F}_2(W)} .$$

Example 5.1. If the repaired item has the same life length distribution as the original then

$$\tau = \mu + \frac{\mu F(W)}{\bar{F}(W)} = \frac{\mu}{\bar{F}(W)}$$

and

$$E(k_1) = c_1 + \frac{c_2 F(W)}{\bar{F}(W)} .$$

If in addition the distribution is exponential with parameter λ, then

$$\tau = \mu e^{\lambda W}$$

and

$$E(k_1) = c_1 - c_2 + c_2 e^{\lambda W} .$$

5.2. The Standard Warranty Policy

In the standard warranty policy the item and all necessary replacements are replaced or repaired free of charge if the failures occur before time W. The replace-repair policy is shown pictorially in Figure 5.0. For notational convenience the equivalent notation of Figure 5.2 will be used to analyze this policy. The analysis will use

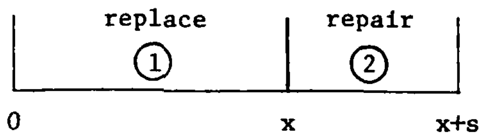


Figure 5.2

An Equivalent Replace-Repair Policy

the same techniques that were used in Section 4.2.

Let $A(x)$ represent the total expected cycle length of the policy depicted in Figure 5.2. Conditioning upon the first time of failure, t, and using the results of Section 4.1,

$$A(x) = \int_0^x t + A(x-t)\, dF_1(t) + \int_{x+s}^{\infty} t dF_1(t)$$

$$+ \int_x^{x+s} t + \mu_2(1 + M_2(s + x - t))\, dF_1(t)$$

$$= \mu_1 + \int_x^{x+s} \mu_2(1 + M_2(s + x - t))\, dF_1(t) + \int_0^x A(x-t)\, dF_1(t) \;.$$

Solving for $A(x)$ via the renewal theorem,

$$A(x) = \mu_1(1 + M_1(x)) + \int_x^{x+s} \mu_2(1 + M_2(s + x - t))\, dF_1(t)$$

$$+ \int_0^x \int_{x-u}^{x+s-u} \mu_2(1 + M_2(s + x - t - u))\, dF_1(t)\, dM_1(u) \; .$$

As in Chapter 4, $\tau = A(W - s)$ so,

$$\tau = \mu_1(1 + M_1(W - s)) + \int_{W-s}^{W} \mu_2(1 - M_2(W - t))\, dF_1(t)$$

$$+ \int_0^{W-s} \int_{W-s-u}^{W-u} \mu_2(1 + M_2(W - t - u))\, dF_1(t)\, dM_1(u) \; .$$

The expected cost to the manufacturer of a cycle or the expected cost per item sold, $E(k_1)$, is found by letting $B(x)$ represent the total expected cost (after time zero) of a cycle.

$$B(x) = \int_0^x c_1 + B(x - t)\, dF_1(t) + \int_x^{x+s} c_2(1 + M_2(x + s - t))\, dF_1(t)$$

$$= c_1F_1(x) + \int_x^{x+s} c_2(1 + M_2(x + s - t))\, dF_1(t) + \int_0^x B(x - t)\, dF_1(t) \; .$$

This is in the form of a renewal equation and can be solved using the renewal theorem.

$$B(x) = c_1 F_1(x) + \int_x^{x+s} c_2(1 + M_2(x + s - t))\, dF_1(t)$$

$$+ \int_0^x c_1 F_1(x - u)\, dM_1(u)$$

$$+ \int_0^x \int_{x-u}^{x+s-u} c_2(1 + M_2(x + s - t - u))\, dF_1(t)\, dM_1(u)$$

$$= c_1 M_1(x) + \int_x^{x+s} c_2(1 + M_2(x + s - t))\, dF_1(t)$$

$$+ \int_0^x \int_{x-u}^{x+s-u} c_2(1 + M_2(x + s - t - u))\, dF_1(t)\, dM_1(u) \;.$$

The total expected cost of a cylce is $E(k_1) = c_1 + B(W - s)$.

The following theorem will be used in Chapter 6 to prove the optimality of certain types of replacement policies.

Theorem 5.2. If $F_1(x)$ is exponential with parameter λ, then $B(x) = c_1\lambda_1 x + h(s)$ where $h(s)$ is a function of s and $M_2(\cdot)$ only.

Proof: If $F_1(x)$ is exponential with parameter λ, then

$$\int_{x-u}^{x+s-u} c_2(1 + M_2(x + s - t - u))\, dF_1(t)$$

$$= \int_{x-u}^{x+s-u} c_2(1 + M_2(x + s - t - u))\, \lambda e^{-\lambda t}\, dt \;.$$

Using the change of variables

$$y = t + u - x = e^{-\lambda x} e^{\lambda u} \int_0^s c_2(1 + M_2(s - y)) \lambda e^{-\lambda y} dy .$$

Substituting into the equation for $B(x)$ and realizing that the above identity also holds for $u = 0$,

$$B(x) = c_1\lambda x + e^{-\lambda x} \int_0^s c_2(1 + M_2(s - y)) \lambda e^{-\lambda y} dy$$

$$+ e^{-\lambda x} \int_0^x e^{\lambda u} \int_0^s c_2(1 + M_2(s - y)) \lambda e^{-\lambda y} dy \lambda du$$

$$= c_1\lambda x + e^{-\lambda x} \int_0^s c_2(1 + M_2(s - y)) \lambda e^{-\lambda y} dy$$

$$+ e^{-\lambda x} \int_0^x \lambda e^{\lambda u} du \int_0^s c_2(1 + M_2(s - y)) \lambda e^{-\lambda y} dy .$$

Noting $e^{-\lambda x} \int_0^x \lambda e^{\lambda u} du = 1 - e^{-\lambda x}$,

$$B(x) = c_1\lambda x + \int_0^s c_2(1 + M_2(s - y) \lambda e^{-\lambda y} dy$$

$$= c_1\lambda x + h(s) . \qquad \boxtimes$$

The above formula for $B(x)$ can be verified virtually by inspection. If $F_1(x)$ is exponential, then the expected number of renewals during the interval $[0, x]$ is λx. The memoryless property of the exponential distribution implies the distribution of the remaining life of the current item at time x is still exponential. Hence, by conditioning upon the first failure after time x, the above formula for $B(x)$ is seen to be true.

<u>Example 5.2</u>. If $F_1(\cdot)$ and $F_2(\cdot)$ are exponential distributions with parameters λ_1 and λ_2 respectively, then τ, the expected time until the consumer will have to pay for a new item, is found from $A(W-s)$. Using the results derived in the proof of Theorem 5.2,

$$\begin{aligned}
\tau &= A(W - s) \\
&= \mu_1(1 + \lambda_1(W - s)) + \int_0^s \mu_2\,(1 + \lambda_2(s - y))\,\lambda_1 e^{-\lambda_1 y}\,dy \\
&= \mu_1 + W - s + \mu_2 F_1(s) + sF_1(s) - \int_0^s \lambda_1 y e^{-\lambda_1 y}\,dy \\
&= \mu_1 + W - s + (\mu_2 + s)\,F_1(s) - \mu_1 + \mu_1 e^{-\lambda_1 s} + s e^{-\lambda_1 s} \\
&= W - s + \mu_2 - \mu_2 e^{-\lambda_1 s} + \mu_1 e^{-\lambda_1 s} .
\end{aligned}$$

The above formula can also be verified by inspection. At time $W - s$, a new item with distribution $F_1(\cdot)$ is operating. It will fail within the next s units of time (i.e., before time W) and be repaired with probability $F_1(s)$. Thus, with probability $F_1(s)$ the remaining expected life at time W is μ_2. If the item at time $W - s$ does not fail during the next s units, then due to the memoryless property of the exponential the expected remaining life is μ_1 and hence,

$$\tau = W + \mu_2 F_1(s) + \mu_1 \bar{F}_1(s) \ .$$

The total expected cost to the manufacturer of a cycle can also be looked at as the total expected cost to the manufacturer per item sold. In either case it is $E(k_1)$. Using Theorem 5.2,

$$E(k_1) = c_1 + c_1\lambda_1(W - s) + \int_0^s c_2(1 + \lambda_2(s - y))\ \lambda_1 e^{-\lambda_1 y}\ dy$$

$$= c_1 + c_1\lambda_1(W - s) + c_2\left(1 - e^{-\lambda_1 s}\right) + c_2\lambda_2 s\left(1 - e^{-\lambda_1 s}\right)$$

$$- c_2\lambda_2 \int_0^s \lambda_1 y e^{-\lambda_1 y}\ dy$$

$$= c_1 + c_1\lambda_1(W - s) + c_2(1 + \lambda_2 s)\ \left(1 - e^{-\lambda_1 s}\right) - c_2\lambda_2\mu_1$$

$$+ c_2\lambda_2\mu_1 e^{-\lambda_1 s} + c_2\lambda_2 s e^{-\lambda_1 s}$$

$$= c_1 + c_1\lambda_1(W - s) + c_2(1 + \lambda_2(s - \mu_1))$$

$$+ (c_2\lambda_2\mu_1 - c_2)\, e^{-\lambda_1 s} .$$

Notice that if $s = 0$ the formula simplifies nicely to $c_1 + c_1\lambda_1 W$, the value derived in Section 4.1. Also, if $c_1 = c_2$ and $\lambda_1 = \lambda_2$ the same value is derived,

$$E(k_1) = c_1 + c_1\lambda_1(W - s) + c_1 + c_1\lambda_1(s - \mu_1)$$

$$+ (c_1\lambda_1\mu_1 - c_1)\, e^{-\lambda_1 s}$$

$$= c_1 + c_1\lambda_1 W - c_1\lambda_1 s + c_1 + c_1\lambda_1 s - c_1$$

$$= c_1 + c_1\lambda_1 W .$$

5.3. The (T, W) Warranty Policy

In the (T, W) warranty policy the item is replaced or repaired free of charge any time the item fails before time W. In addition, a new warranty of length max (W - t, T) is issued with the replaced or repaired item. Since the new warranty is at least of length T, the only replace-repair policy that needs to be considered is as depicted below in Figure 5.3.1.

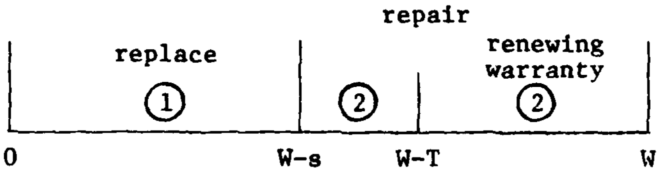


Figure 5.3.1

The Replace-Repair Policy for the (T, W) Warranty

For notational convenience, however, the equivalent policy in Figure 5.3.2 will be analyzed.

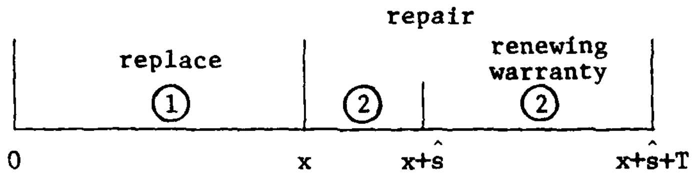


Figure 5.3.2

An Equivalent Policy

Let $A(x)$ once again represent the total expected cycle length of the above policy. Conditioning upon the first failure time, t, and using the results of Section 4.2

$$A(x) = \int_0^x t + A(x-t)\, dF_1(t) + \int_{x+\hat{s}+T}^{\infty} t dF_1(t)$$

$$+ \int_x^{x+\hat{s}} t + \alpha(x+\hat{s}-t) dF_1(t) + \int_{x+\hat{s}}^{x+\hat{s}+T} t + \frac{\mu_2}{\bar{F}_2(T)}\, dF_1(t)$$

where $a(\cdot)$ is found from Section 4.2 to be

$$a(z) = \mu_2 + \frac{\mu_2}{\bar{F}_2(T)} (F_2(z + T) - F_2(z)) + \mu_2 M_2(z)$$

$$+ \frac{\mu_2}{\bar{F}_2(T)} \int_0^z (z + T - u) - F(z - u)\ dM_2(u)\ .$$

Letting

$$D(x, \hat{s}) = \int_x^{x+\hat{s}} a(x + \hat{s} - t)\ dF_1(t) + \int_{x+\hat{s}}^{x+\hat{s}+T} \frac{\mu_2}{\bar{F}_2(T)}\ dF_1(t)\ ,$$

$$A(x) = \int_0^\infty t dF_1(t) + D(x, \hat{s}) + \int_0^x A(x - t)\ dF_1(t)$$

$$= \mu_1 + D(x, \hat{s}) + \int_0^x A(x - t)\ dF_1(t)\ .$$

By use of the renewal theorem

$$A(x) = \mu_1(1 + M_1(x)) + D(x, \hat{s}) + \int_0^x D(x - t, \hat{s})\ dM_1(t)\ .$$

τ, the expected cycle length is found by reverting back to the notation of Figure 5.3.1.

$$\tau = \mu_1(1 + M_1(W - s)) + D(W - s, s - T) + \int_0^{W-s} D(W - s - t, s - T)\ dM_1(t)\ .$$

The expected cost to the manufacturer of a cycle, $E[k_1]$ is similarly found from

$$B(x) = \int_0^x c_1 + B(x - t)\, dF_1(t) + \int_x^{x+\hat{s}} c_2 + E(x + \hat{s} - t)\, dF_1(t)$$

$$+ \int_{x+\hat{s}}^{x+\hat{s}+T} \frac{c_2}{\bar{\bar{F}}_2(T)}\, dF_1(t)$$

where $B(x)$ represents the total expected cost (not counting the cost at time zero) to the manufacturer of a cycle and $E(\cdot)$ is (from Section 4.2)

$$E(z) = c_2M_2(z) + \frac{c_2}{\bar{F}_2(T)} (F_2(z + T) - F_2(z))$$

$$+ \frac{c_2}{\bar{\bar{F}}_2(T)} \int_0^z (F_2(z + T - u) - F_2(z - u))\, dM_2(u) .$$

Letting

$$\hat{D}(x, \hat{s}) = c_1F_1(x) + c_2(F_1(x - \hat{s}) - F_1(x))$$

$$+ \frac{c_2}{\bar{\bar{F}}_2(T)} (F_1(x + \hat{s} + T) - F_1(x - \hat{s}))$$

$$+ \int_x^{x+\hat{s}} E(x + \hat{s} - t)\, dF_1(t) ,$$

and using the renewal theorem,

$$B(x) = \hat{D}(x, \hat{s}) + \int_0^x \hat{D}(x - t, \hat{s})\, dM_1(t) \, .$$

The expected cost per cycle is thus

$$E[k_1] = c_1 + \hat{D}(W - s, s - T) + \int_0^{W-s} \hat{D}(W - s - t, s - T)\, dM_1(t) \, .$$

<u>Theorem 5.3</u>. If $F_1(\cdot)$ is exponential with parameter λ, then $B(x) = c_1\lambda_1 x + h(s, T)$ where $h(s, T)$ is independent of x.

<u>Proof</u>: If $F_1(\cdot)$ is exponential with parameter λ, then by use of the memoryless property of the exponential

$$B(x) = c_1\lambda_1 x + h(s, T)$$

where $h_1(s, T)$ represents the total expected cost of the policy in Figure 5.3.3.

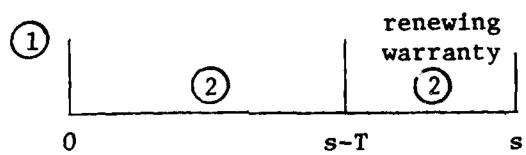


Figure 5.3.3

As can be seen by the figure, $h(s, T)$ is independent of x and, in fact, is a function of s, T, and $F_2(\cdot)$ only. ⊠

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Example 5.3. If $F_1(\cdot)$ and $F_2(\cdot)$ are exponential distributions with parameters λ_1 and λ_2 respectively, then τ, the expected time until the consumer will have to pay for a new item, is found from $A(x)$. As in Theorem 5.3, this simplifies to x plus the total expected length of the policy depicted in Figure 5.3.3. Using the results of Example 4.2,

$$A(x) = x + \int_0^{s-T} \left(s - T + \mu_2 e^{\lambda_2 T}\right) \lambda_1 e^{-\lambda_1 y}\, dy$$

$$+ \int_{s-T}^{s} y + \mu_2 e^{\lambda_2 T}\, \lambda_1 e^{-\lambda_1 y}\, dy + \int_s^{\infty} y \lambda_1 e^{-\lambda_1 y}\, dy$$

$$= x + \left(s - T + \mu_2 e^{\lambda_2 T}\right)\left(1 - e^{-\lambda_1 (s-T)}\right)$$

$$+ \mu_2 e^{\lambda_2 T}\left(e^{-\lambda_1 (s-T)} - e^{-\lambda_1 s}\right)$$

$$+ \mu_1\left(e^{-\lambda_1 (s-T)} - e^{-\lambda_1 s}\right)$$

$$- s e^{-\lambda_1 s} + (s - T)\, e^{-\lambda_1 (s-T)} + s e^{-\lambda_1 s} + \mu_1 e^{-\lambda_1 s}$$

$$= x + s - T + \mu_2 e^{\lambda_2 T} + e^{-\lambda_1 s}\left[\mu_1 e^{\lambda_1 T} - \mu_2 e^{\lambda_2 T}\right].$$

τ is found from $A(W - s)$.

The total expected cost, to the manufacturer, of a cycle is found from Theorem 5.3 and Example 4.2

$$E(k_1) = c_1 + c_1\lambda_1 x + \int_0^{s-T} c_2\lambda_2(s - T - t) + c_2 e^{\lambda_2 T} \, dF_1(t)$$

$$+ \int_{s-T}^{s} c_2 e^{\lambda_2 T} \, dF_1(t)$$

$$= c_1 + c_1\lambda_1 x + c_2\lambda_2(s - T)\left(1 - e^{-\lambda_1(s-T)}\right)$$

$$+ c_2 e^{\lambda_2 T}\left(1 - e^{-\lambda_1 s}\right) - c_2\lambda_2\left(\mu_1 - \mu_1 e^{-\lambda_1(s-T)} - (s-T)\, e^{-\lambda_1(s-T)}\right)$$

$$= c_1 + c_1\lambda_1 x + c_2\lambda_2(s - T - \mu_1) + c_2 e^{\lambda_2 T}$$

$$+ c_2 e^{-\lambda_1 s}\left(\lambda_2\mu_1 e^{\lambda_1 T} - e^{\lambda_2 T}\right) .$$

To get comparable values use $x = W - s$.

5.4. Optimal Replace-Repair Policies for Exponential Distributions

Given the fact that the manufacturer wishes to have a replace-repair policy and assuming that the manufacturer wishes to maximize expected profit per item sold, the optimal policy for the exponential case can be found. This optimal policy will be derived for both the standard warranty policy and the (T, W) warranty policy. In both cases the optimal policy will be derived by finding s^*, the value of s that maximizes the expected profit per item sold.

Consider the standard warranty policy as depicted below in Figure 5.4.1. Assume replaced and repaired items are all distributed exponentially with parameters λ_1 and λ_2 and costs c_1 and c_2 respectively.

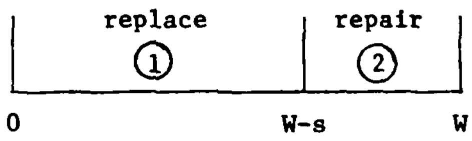


Figure 5.4.1

Standard Warranty Policy

Theorem 5.4.1. If $c_1 > c_2$ and $c_1\lambda_1 < c_2\lambda_2$, then the value of s that maximizes the expected profit per item sold for the standard warranty policy is

$$s^* = \min\left(-\mu_1 \ln\left[\frac{c_1\lambda_1 - c_2\lambda_2}{c_2\lambda_1 - c_2\lambda_2}\right], W\right).$$

Furthermore, s^* is positive.

Proof: The expected profit per item sold is $c_0 - E(k_1)$. Thus, maximizing profit is equivalent to minimizing $E(k_1)$. From Example 5.2

$$E(k_1) = c_1 + c_1\lambda_1(W - s) + c_2(1 + \lambda_2(s - \mu_1))$$

$$+ (c_2\lambda_2\mu_1 - c_2)\, e^{-\lambda_1 s}$$

$E(k_1)$ is convex in s if $\dfrac{\partial^2 E(k_1)}{\partial s^2} \geq 0$ for all s.

$$\frac{\partial E(k_1)}{\partial s} = - c_1\lambda_1 + c_2\lambda_2 + (c_2\lambda_1 - c_2\lambda_2)\, e^{-\lambda_1 s} .$$

$$\frac{\partial E(k_1)}{\partial s^2} = \lambda_1(c_2(\lambda_2 - \lambda_1)) .$$

Thus, $E(k_1)$ is strictly convex in s iff $\lambda_2 > \lambda_1$. Note that this condition is a result of the hypotheses that $c_1 > c_2$ and $c_1\lambda_1 < c_2\lambda_2$. Since $E(k_1)$ is convex the value of s that minimizes $E(k_1)$ is that value of s that sets $\dfrac{\partial E(k_1)}{\partial s} = 0$,

$$- c_1\lambda_1 + c_2\lambda_2 + (c_2\lambda_1 - c_2\lambda_2)\, e^{-\lambda_1 \hat{s}} = 0 .$$

Solving for $\hat{s}$,

$$\hat{s} = - \mu_1 \ln\left[\frac{c_1\lambda_1 - c_2\lambda_2}{c_2\lambda_1 - c_2\lambda_2}\right]$$

$$= \mu_1 \ln\left[\frac{c_2(\lambda_2 - \lambda_1)}{c_2\lambda_2 - c_1\lambda_1}\right].$$

The condition $\lambda_2 > \lambda_1$ implies the necessity of $c_2\lambda_2 > c_1\lambda_1$ because otherwise the logarithm would be of a negative number. In order that $\hat{s}$ be positive it is further necessary that

$$c_2(\lambda_2 - \lambda_1) > c_2\lambda_2 - c_1\lambda_1 \qquad \text{or,}$$

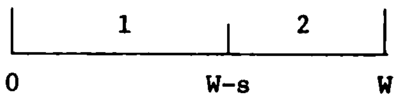
$$c_2 < c_1 \ .$$

Thus, the necessary conditions for $\hat{s}$ to be positive and the point of a global minimum are $c_2 < c_1$ and $c_2\lambda_2 > c_1\lambda_1$. The fact that $E(k_1)$ is convex insures that if $\hat{s} > W$ then the minimum feasible cost occurs when $s = W$. ☒

Using the results of Theorem 5.4.1 it is possible to graph the values of s^* for various values of c_1, c_2, λ_1, and λ_2. A plot of these values is given in Figure 5.4.2. Without loss of generality temporal and monetary units have been adjusted so that $c_1 = 1$ monetary unit and $\mu_1 = 1/\lambda_1 = 1$ time unit. The equation for $\hat{s}$ simplifies to

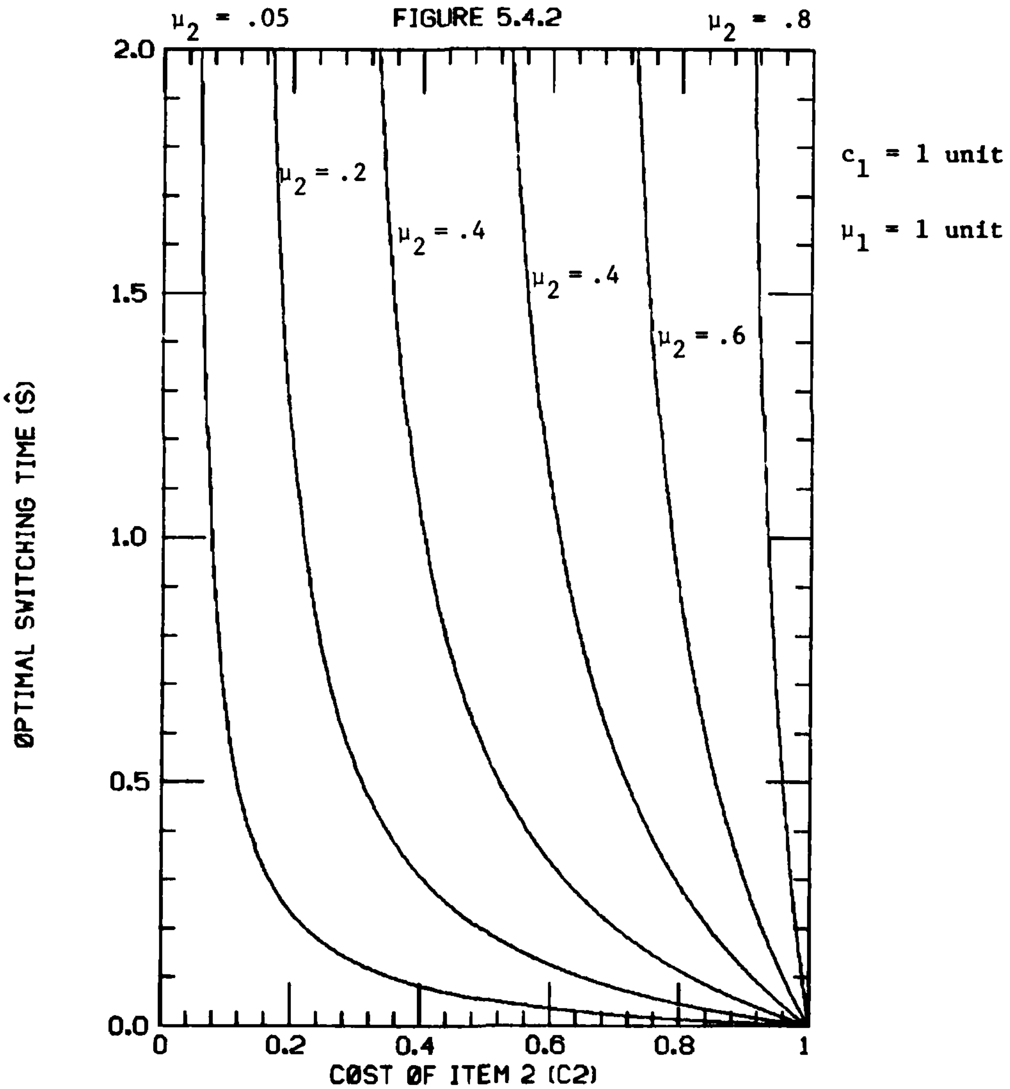
$$\hat{s} = \ln \frac{c_2(\lambda_2 - 1)}{c_2\lambda_2 - 1}$$

with $c_2 < 1$ and $c_2\lambda_2 > 1$ or, $\mu_2 < c_2 < 1$.



switching policy

FIGURE 5.4.2



Optimal Switching Times - Standard Warranty

A similar analysis can be performed for the (T, W) warranty policy. The notation used will be that of Figure 5.4.3, with x

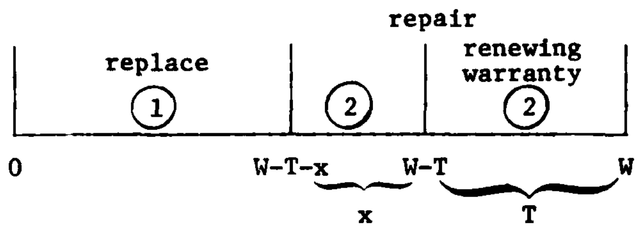


Figure 5.4.3

representing the amount of time before the renewing warranty period begins after which the manufacturer should begin repairing in lieu of replacing. Under the previously defined notation $s^* = x^* + T$.

The total expected cost to the manufacturer per item sold is $E(k_1)$. From Section 5.3, using the appropriate change of notation,

$$E(k_1) = c_1 + c_1\lambda_1(W - T - x) + c_2\lambda_2(x - \mu_1) + c_2 e^{\lambda_2 T}$$

$$+ c_2 e^{-\lambda_1 (x+T)}\left(\lambda_2 \mu_1 e^{\lambda_1 T} - e^{\lambda_2 T}\right) .$$

The optimal replace-repair policy for this warranty will be derived in Theorems 5.4.2 and 5.4.3 (depending upon the value of T). Throughout the theorems and proofs, the "feasible values of $E(k_1)$" will refer to $x \in [0, W - T]$. The other feasible policy is the replace-only policy which will be considered when appropriate.

Theorem 5.4.2. If

(1) $c_2 \le c_1$,

(2) $c_1\lambda_1 < c_2\lambda_2$ and

(3) $T \le \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2}$,

then there exists a nonnegative value of x, x^*, that maximizes expected profit per unit sold for the replace-repair policy of the (T, W) warranty as depicted in Figure 5.4.3. Furthermore, the value of x^* is

$$x^* = \min\left(W - T,\ \mu_1 \ln \left[\frac{c_2\lambda_2-c_2\lambda_1 e^{(\lambda_2-\lambda_1)T}}{c_2\lambda_2-c_1\lambda_1}\right]\right).$$

To prove this theorem three short lemmas are needed.

Lemma 1: $\lambda_2 > \lambda_1$.

Proof: By hypothesis

$$c_1 \ge c_2 > 0 \quad \text{and}$$
$$c_2\lambda_2 > c_1\lambda_1 > 0 .$$

Thus,

$$\lambda_2 > \frac{c_1}{c_2}\lambda_1 > \lambda_1 . \qquad \boxtimes$$

Lemma 2:

$$0 < \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} < \frac{1}{\lambda_2-\lambda_1} \ln \frac{\lambda_2}{\lambda_1} .$$

Proof:

$$0 < c_1\lambda_1 < c_2\lambda_2 \qquad \text{by hypothesis} .$$

Thus,

$$0 < \frac{c_1}{c_2} < \frac{\lambda_2}{\lambda_1} .$$

Using the monotonicity of the natural logarithm and Lemma 1,

$$0 < \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} < \frac{1}{\lambda_2-\lambda_1} \ln \frac{\lambda_2}{\lambda_1} . \qquad \boxtimes$$

Lemma 3: If

$$T \leq (>) \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} ,$$

then the expected cost per item sold when $x = 0$ is less than or equal to (greater than) the expected cost per item sold for the replace-only policy.

Proof: The $E(k_1)$ when $x = 0$ is

$$E(k_1)\big|_{x=0} = c_1 + c_1\lambda_1(W - T) - c_2\lambda_2\mu_2 + c_2 e^{\lambda_2 T}$$

$$+ c_2 e^{-\lambda_1 T}\left(\lambda_2\mu_1 e^{\lambda_1 T} - e^{\lambda_2 T}\right)$$

$$= c_1 + c_1\lambda_1(W - T) + c_2 e^{\lambda_2 T} - c_2 e^{(\lambda_2 - \lambda_1)T} .$$

The expected cost with no option of repairing is found in Section 4.3

$$c_1\lambda_1(W - T) + c_1 e^{\lambda_1 T} .$$

Thus, the $x = 0$ policy is "better than" or equal to the replace-only policy iff

$$c_1 + c_2 e^{\lambda_2 T} - c_2 e^{(\lambda_2 - \lambda_1)T} \leq c_1 e^{\lambda_1 T} \qquad \text{iff}$$

$$c_2 e^{\lambda_2 T}\left(1 - e^{-\lambda_1 T}\right) \leq c_1 e^{\lambda_1 T}\left(1 - e^{-\lambda_1 T}\right) \qquad \text{iff}$$

$$c_2 e^{\lambda_2 T} \leq c_1 e^{\lambda_1 T} .$$

Taking logarithms and rearranging terms via Lemma 1, this is seen to be equivalent to

$$T \leq \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} ,$$

which is the hypothesis. ☒

Proof of Theorem 5.4.2. The expected profit per item sold is c_0 minus the expected cost to the manufacturer of a cycle or $c_0 - E(k_1)$. Thus, maximizing profit is equivalent to minimizing $E(k_1)$. The first thing to note is that by Lemma 3 and the third hypothesis, the expected cost of the (W, T) warranty with no option of repairing (i.e., the replace-only policy) is greater than or equal to the expected cost with the option of repairing. Thus, the optimal policy includes repairs.

To find the optimal value of x it is necessary to take the derivatives of the equation for $E(k_1)$.

$$\frac{\partial[E(k_1)]}{\partial x} = c_2\lambda_2 - c_1\lambda_1 + c_2e^{-\lambda_1 x}\left(\lambda_1 e^{(\lambda_2-\lambda_1)T} - \lambda_2\right) .$$

$$\frac{\partial^2[E(k_1)]}{\partial x^2} = \lambda_1 c_2 e^{-\lambda_1 x}\left(\lambda_2 - \lambda_1 e^{(\lambda_2-\lambda_1)T}\right) .$$

The formula for $E(k_1)$ is convex in x since

$$\frac{\partial^2}{\partial x^2}\,[E(k_1)] \geq 0 \qquad \text{iff}$$

$$\lambda_2 - \lambda_1 e^{(\lambda_2-\lambda_1)T} \geq 0 \qquad \text{iff}$$

$$T \leq \frac{1}{\lambda_2-\lambda_1} \ln \frac{\lambda_2}{\lambda_1} ,$$

which is true by Lemma 3 and the third hypothesis.

The point $\hat{x}$ at which the first derivative is zero is, thus, the optimal point, if feasible, and is found from

$$0 = c_2\lambda_2 - c_1\lambda_1 + e^{-\lambda_1\hat{x}}\left(c_2\lambda_1 e^{(\lambda_2-\lambda_1)T} - c_2\lambda_2\right).$$

Solving for $\hat{x}$,

$$\hat{x} = \mu_1 \ln\left[\frac{c_2\lambda_2 - c_2\lambda_1 e^{(\lambda_2-\lambda_1)T}}{c_2\lambda_2 - c_1\lambda_1}\right].$$

$\hat{x}$ is nonnegative if the numerator is greater than or equal to the denominator (the denominator is positive by hypothesis) or if

$$c_2 e^{(\lambda_2-\lambda_1)T} \leq c_1 .$$

Rearranging terms and using Lemma 1, this is equivalent to

$$T \leq \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} ,$$

the third hypothesis. Thus, $\hat{x}$ is nonnegative.

As in the proof of Theorem 5.4.1, $E(k_1)$ convex in x implies that if $\hat{x} > W - T$, then the optimal feasible value of $E(k_1)$ occurs at $x = W - T$. ☒

<u>Theorem 5.4.3</u>. Under the hypothesis of Theorem 5.4.2 with

$$(3') \qquad T > \frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2}$$

the optimal replace-repair policy is to always replace.

<u>Proof</u>: If

$$\frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} < T < \frac{1}{\lambda_2-\lambda_1} \ln \frac{\lambda_2}{\lambda_1}$$

then $E(k_1)$ is convex in x by the proof of Theorem 5.2. The value of x that minimizes $E(k_1)$ is still

$$x = \mu_1 \ln \frac{c_2\lambda_2-c_2\lambda_1 e^{(\lambda_2-\lambda_1)T}}{c_2\lambda_2-c_1\lambda_1} .$$

By the above assumption,

$$\frac{1}{\lambda_2-\lambda_1} \ln \frac{c_1}{c_2} < T < \frac{1}{\lambda_2-\lambda_1} \ln \frac{\lambda_2}{\lambda_1}$$

which implies by Lemma 1

$$\ln \frac{c_1}{c_2} < (\lambda_2-\lambda_1)\ T < \ln \frac{\lambda_2}{\lambda_1} \ .$$

Exponentiating each term

$$\frac{c_1}{c_2} < e^{(\lambda_2-\lambda_1)T} < \frac{\lambda_2}{\lambda_1} \qquad \text{or,}$$

$$c_1\lambda_1 < c_2\lambda_1 e^{(\lambda_2-\lambda_1)T} < \lambda_2 c_2 \ .$$

Subtracting $c_2\lambda_2$ and multiplying by minus one,

$$c_2\lambda_2 - c_1\lambda_1 > c_2\lambda_2 - c_2\lambda_1 e^{(\lambda_2-\lambda_1)T} > 0 \ .$$

Finally, dividing through by $c_2\lambda_2 - c_1\lambda_1$ and taking the logarithm

$$\ln \frac{c_2\lambda_2 - c_2\lambda_1 e^{(\lambda_2-\lambda_1)T}}{c_2\lambda_2 - c_1\lambda_1} < 0 \ .$$

This implies $\hat{x} < 0$ (from the formula for $\hat{x}$) which in turn implies the optimal feasible value occurs at $x^* = 0$, due to the convexity of $E(k_1)$.

However, Lemma 3 states that if

$$T > \frac{1}{\lambda_2 - \lambda_1} \ln \frac{c_1}{c_2}$$

then the always-replace policy is superior to the $x = 0$ policy. Thus, if

$$\frac{1}{\lambda_2 - \lambda_1} \ln \frac{c_1}{c_2} < T < \frac{1}{\lambda_2 - \lambda_1} \ln \frac{\lambda_2}{\lambda_1} ,$$

the optimal replace-repair policy is to always replace.

If

$$T \geq \frac{1}{\lambda_2 - \lambda_1} \ln \frac{\lambda_2}{\lambda_1} \qquad \text{then,}$$

$$\frac{\partial(E(k_1))}{\partial x} = c_2\lambda_2 - c_1\lambda_1 + c_2 e^{-\lambda_1 x} \left(\lambda_1 e^{(\lambda_2 - \lambda_1)T} - \lambda_2\right)$$

is greater than or equal to zero for all x. Thus, the minimum feasible cost occurs at the minimum value of x or, at $x = 0$. By Lemma 2 and Lemma 3 once again, the always-replace police is superior to the $x = 0$ policy.

Hence, combining the two cases, if

$$T > \frac{1}{\lambda_2 - \lambda_1} \ln \frac{c_1}{c_2}$$

then always-replace is optimal. ⊠

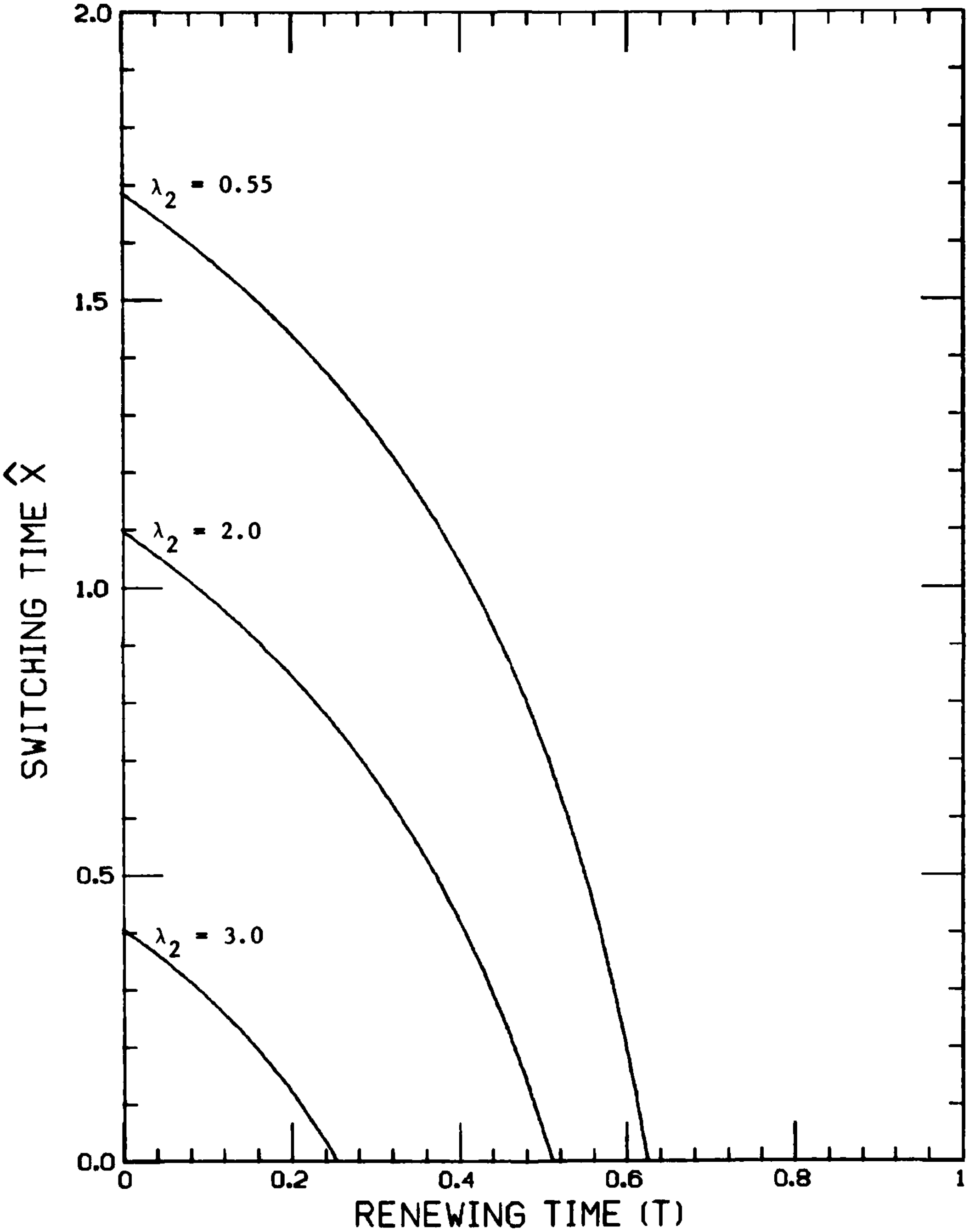
Figure 5.4.4 is a plot of $\hat{x}$ vs T. By use of appropriate scaling $\mu_1 = 1$ and $c_1 = 1$. This simplifies the equation for $\hat{x}$ to

$$\hat{x} = \ln\left[\frac{c_2\lambda_2 - c_2 e^{(\lambda_2-1)T}}{c_2\lambda_2 - 1}\right].$$

By assigning c_2 a reasonable value, 0.6 for instance, $\hat{x}$ can be plotted as a function of λ_2 and T. In this figure λ_2 takes on three values 0.55, 2.0 and 3.0.

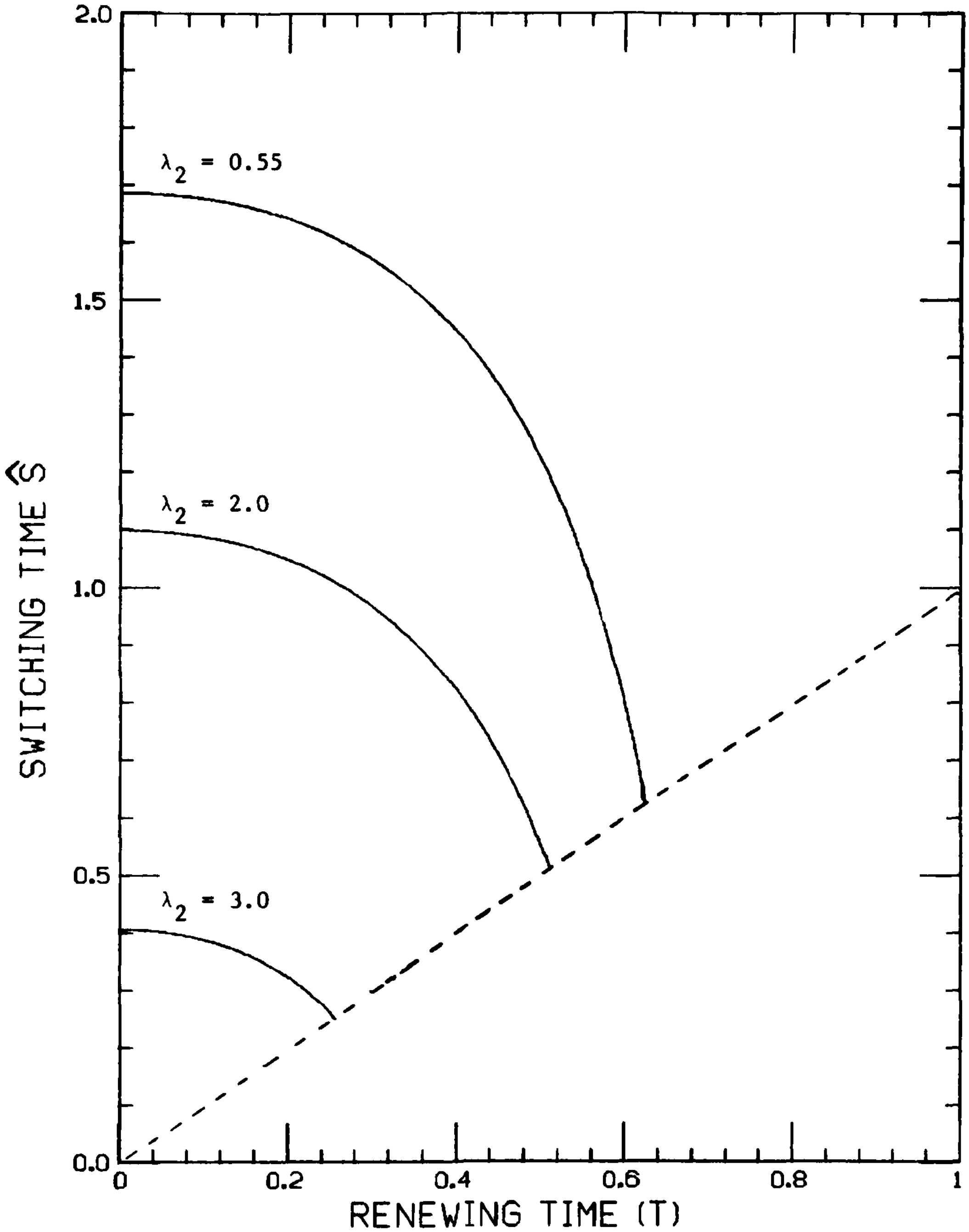
Figure 5.4.5 is a graph of the same values but with the s notation. Recall that in the s notation whenever an item fails under warranty it is replaced if the remaining warranty length is greater than s and repaired otherwise.

FIGURE 5.4.4



Optimal Switching Time $\hat{x}$ vs. T in (T, W) Warranty

FIGURE 5.4.5



Optimal Switching Time $\hat{s}$ vs. T in (T, W) Warranty

CHAPTER 6

OPTIMAL REPAIR POLICIES

In the previous chapter replace-repair policies were discussed and analyzed in detail. In particular, the optimal replace-repair policy was derived for various warranties under the assumption of exponential life lengths. A natural question that arises from this analysis is: "When is a replace-repair policy optimal?" The answer is not a simple one, even when it is limited strictly to the standard warranty, due to the intricacies of the many different possible life length distributions.

Throughout this chapter it will be assumed that when an item under warranty fails, the manufacturer has two options: replace the item with a new item at a cost c_1, or repair the item at a reduced cost $c_2 < c_1$.* It will also be assumed that the warranty policy is the standard warranty (as defined in Chapter 2). Optimal policies will be derived by the use of continuous dynamic programming techniques.

Whenever an item under warranty fails the manufacturer is faced with the decision of whether to repair or replace the item. If the cost of repairing is greater than the cost of replacing, the manufacturer will usually choose to replace. If not, the manufacturer must make some estimate of the trade-offs that exist between the cheaper cost of repairing and the possibly larger probability of failure during the

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*In general one might also with to assume that the repaired item is degraded in the sense that μ_2, the expected life of the repaired item, is less than μ_1, the expected life of a new item. However, this assumption is not necessary for the results that follow.

remaining warranty period. (If the failure distribution of the repaired item is stochastically less than that of a new item for the remaining period of the warranty, then clearly the manufacturer should repair instead of replace.)

Whenever the remaining warranty length is small (for instance, no remaining warranty) and $c_2 < c_1$, the manufacturer should repair. Using the notation of Chapter 5 let $F_1(\cdot)$ represent the failure distribution of a new or replaced item, $F_2(\cdot)$ the failure distribution of a repaired item and s the amount of time before the end of the warranty that one repairs instead of replacing.

<u>Theorem 6.1</u>. If $c_2 < c_1$ and $F_1(0) = F_2(0) = 0$, then there exists a replace-repair policy with nonzero s that is at least as good as the replace-only policy in the sense of minimizing expected cost.

<u>Proof</u>. Let $M_1(t)$ and $M_2(t)$ be the renewal functions associated with $F_1(t)$ and $F_2(t)$ respectively. $M_1(t)$ and $M_2(t)$ are thus bounded and nondecreasing in t. Also, by the bounded convergence theorem $M_1(t)$ and $M_2(t)$ are right continuous. Hence, for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|t - 0| < \delta \quad \text{implies} \quad |M_2(t) - M_2(0)| < \varepsilon \ .$$

In particular, if $\varepsilon = (c_1 - c_2)/c_2$ (greater than zero by hypothesis) then there exists a δ_0 such that

$$t < \delta_0 \quad \text{implies} \quad M_2(t) < \frac{c_1 - c_2}{c_2} .$$

If a failure occurs during the interval $(W - \delta_0, W]$ then the total expected cost of the replace-only policy is

$$c_1 + c_1 M_1(t)$$

where t is the remaining time in the warranty. Likewise, the total expected cost of a repair-only policy is $c_2 + c_2 M_2(t)$.

Recalling that $t < \delta_0$

$$\begin{aligned} c_2 + c_2 M_2(t) &< c_2 + \frac{c_2(c_1 - c_2)}{c_2} \\ &= c_1 \\ &\leq c_1 + c_1 M_1(t) . \end{aligned}$$

Thus, if a failure occurs during the interval $(W - \delta_0, W]$ it is better to repair than it is to replace and by the nature of the total expected cost calculation, the replace-repair policy with $s = \delta_0$ is at least as good as the replace-only policy. ⊠

The reason for equality in the above theorem is that under certain $F_1(\cdot)$ failure distributions it would be impossible to have a failure during the interval $(W - \delta_0, W]$. For instance, if $F_1(\cdot)$ were degenerate at point $2W$ then no failures would ever occur during the warranty period and the total expected cost to the manufacturer would be c_1 for any repair policy.

Now that it has been shown that the replace-repair policy is at least as good as a replace-only policy, the next question is: when is a replace-repair policy not optimal? The following example reveals some of the difficulties in this problem. It should be noted that the difficulties are not simply a result of the degenerate distribution. In fact, if the distribution were "smoothed" just enough to remove the degeneracies a similar optimal policy would still occur.

<u>Example 6.1</u>. A Replace-Repair-Replace-Repair Optimal Policy

Let $F_2(\cdot)$ be such that the probability of failure at time 3 is 1/2 and the probability of failure at time 6 is also 1/2, as depicted below.

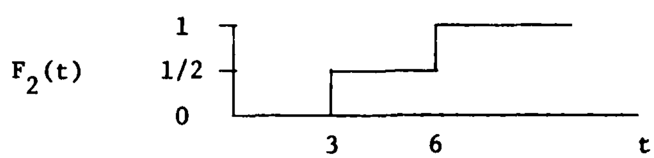

Figure 6.1. $F_2(t)$ for Example 6.1

The expected life length of a repaired item is thus

$$\mu_2 = (6 + 3)/2 = 4.5 \ .$$

Let $F_1(\cdot)$ be exponential with parameter $\mu = 10$. Let $c_1 = 1$, $c_2 = 8/9$ and $W = 7$.

Using the techniques of continuous dynamic programming (that is, working backwards), one can see that if a failure occurs during the interval $(4,7]$ then the total expected cost of replacement is at least 1 and the total cost of repair (not expected total cost since no failure can occur for at least 3 time units) is $8/9$.

During the interval $(1,4]$ the total expected cost of repairing is

$$\frac{8}{9} + \frac{1}{2}(0) + \frac{1}{2}(\frac{8}{9}) = \frac{12}{9} = 1.33 .$$

The total expected cost of replacing is

$$1 + \frac{4-t}{10} + \int_0^3 \frac{8}{9} \lambda e^{-\lambda t} = 1 + \frac{4-t}{10} + \frac{8}{9} (.26) = 1.23 + \frac{4-t}{10} .$$

Thus, during the interval

$(1,2.97]$	it is cheaper to repair,
$(2.97,4]$	it is cheaper to replace,
$(4,7]$	it is cheaper to repair .

During the interval $(0,1]$ the total expected cost to repair is at least

$$\frac{8}{9} + \frac{1}{2} (\frac{8}{9}) + \frac{1}{2} (1.23) = 1.95$$

while the total cost to replace is at most

$$1 + \frac{7-t}{10} \leq 1.7 \ .$$

Thus, the optimal policy is as depicted in Figure 6.2.

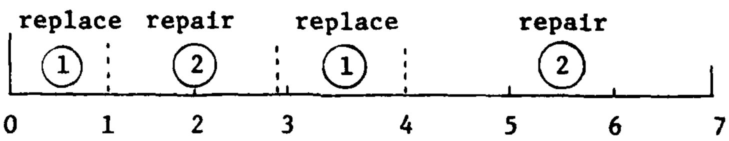

Figure 6.2. Optimal Policy for Example 6.1

A point of interest in this example is that not only is $c_1 > c_2$ (requiring that an optimal repair policy end in repairs as per Theorem 6.1) but μ_2 is also less than μ_1.

One should also note that the equation to solve for the optimal policy is

$$V^*(t) = \min_{i=1,2} c_i + \int_0^t c_{j^*(t-x)} + V^*(t-x)\, dF_i(x),$$

where $j^*(x) \in \{1,2\}$ is the optimal policy determined by the above minimum cost calculation and $V^*(\cdot)$ is the expected cost of that policy. In the example

$$j^*(x) = \begin{cases} 2 & 0 \leq x < 3 \\ 1 & 3 \leq x < 4.03 \\ 2 & 4.03 \leq x < 6 \\ 1 & 6 \leq x < 7 \ . \end{cases}$$

At time zero, $j^*(0)$ is always equal to 2 since $c_2 < c_1$ and by Theorem 6.1 there always exists a $\delta > 0$ such that $j^*(x) = 2$ for all $x \in [0,\delta)$. Hence, the first "switchover point", or point at which the policy changes, is when the two equations

$$c_1 + \int_0^t c_2 + c_2M_2(t - x)\, dF_1(x) \qquad (1)$$

and

$$c_2 + \int_0^t c_2 + c_2M_2(t - x)\, dF_2(x) \qquad (2)$$

$$= c_2(1 + M_2(t))$$

cross over.

In the example given these two equations happen to cross not only once but three times in the interval [0,7]. The three values of t at which these crossings occur are s = 3, 4.01 and 6. These correspond to the points 4, 2.99 and 1 respectively in Figure 6.2*.

Although any subsequent crossings of the two equations do not necessarily reflect switching points, it may still be useful for the manufacturer to know when there exists a unique point s during the warranty period W where the difference of the two equations switches signs. The following two theorems explore conditions under which this is true.

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*It is a bit of a coincidence that the equation crossings at 4.01 and 6 correspond almost exactly to the switching times of 7-4.01 = 2.99 and 7-6 = 1. However, as discussed in the text, it is no coincidence that the point 7-3 = 4 corresponds to a switching time.

Theorem 6.2. If $c_1 > c_2$ and

$$dF_1(x) < dF_2(x) \qquad \forall x \in [0,W]$$

then the two equations

$$c_1 + \int_0^t c_2 + c_2 M_2(t - x)\, dF_1(x) \tag{1}$$

and

$$c_2 + \int_0^t c_2 + c_2 M_2(t - x)\, dF_2(x) \tag{2}$$

cross at most once in the interval $[0,W]$.

Proof. Consider the difference, (2) minus (1)

$$c_2 - c_1 + \int_0^t c_2(1 + M_2(t - x))[dF_2(x) - dF_1(x)] \; .$$

At time $t = 0$ this difference is negative since $c_1 > c_2$. If $dF_2(x) > dF_1(x)$ for all $x \in [0,W]$ then

$$\begin{aligned} \frac{d}{dt} \int_0^t & c_2(1 + M_2(t - x))[dF_2(x) - dF_1(x)] \\ &= c_2[dF_2(t) - dF_1(t)] \\ &\quad + \int_0^t \frac{d}{dt} M_2(t - x)[dF_2(x) - dF_1(x)] \\ &> 0 \end{aligned}$$

since $M_2(t)$ is increasing in t. Thus, there is at most one point $s \in [0,W]$ where the above difference can be equal to zero and, hence, Theorem 6.2 is proved. ⊠

The above theorem is probably not very useful for large W due to the restriction that

$$dF(x) > dF_1(x) \qquad \forall x \in [0,W] \ .$$

Note that this condition cannot hold for all x because

$$\int_0^\infty dF_2(x) = \int_0^\infty dF_1(x) = 1 \ .$$

Another interesting and perhaps more useful theorem can be proved if one assumes that $F_2(\cdot)$ is an exponential distribution. The method of proof will be the same as was used in Theorem 6.2, that is, proving the difference is increasing at t.

Theorem 6.3. If $c_1 > c_2$, $F_2(\cdot)$ is an exponential distribution and $F_1(\cdot)$ has a decreasing failure rate (DFR) distribution with density $f_1(\cdot)$, then the two equations (1) and (2) of Theorem 6.2 cross at most once.

Proof. If $F_2(\cdot)$ is distributed exponentially with parameter λ_2 then

$$M_2(t - x) = \lambda_2(t - x) .$$

Equation (2) minus Equation (1) thus simplifies to

$$c_2 - c_1 + \int_0^t c_2 + c_2\lambda_2(t - x)[dF_1(x) - dF_2(x)]$$

$$= c_2 - c_1 + c_2[F_2(t) - F_1(t)] + c_2\lambda_2 t[F_2(t) - F_1(t)]$$

$$- \int_0^t c_2\lambda_2 \; xdF_2(x) + \int_0^t c_2\lambda_2 \; xdF_1(x) .$$

Notice that at $t = 0$ the above equation is $c_2 - c_1$, less than zero by hypothesis.

To find if this equation is increasing in t one can take the partial derivative with respect to t and see if it is positive. Recalling the assumption that $F_1(\cdot)$ has a density,

$$\frac{\partial}{\partial t}[(2) - (1)] = c_2[f_2(t) - f_1(t)] + c_2\lambda_2 t[f_2(t) - f_1(t)]$$

$$+ c_2\lambda_2[F_2(t) - F_1(t)] - c_2\lambda_2 tf_2(t)$$

$$+ c_2\lambda_2 tf_1(t)$$

$$= c_2\lambda_2 e^{-\lambda_2 t} + c_2\lambda_2 - c_2\lambda_2 e^{-\lambda_2 t} - c_2 f_1(t) - c_2\lambda_2 F_1(t)$$

$$= c_2\lambda_2\bar{F}_1(t) - c_2 f_1(t) .$$

Thus, the partial with respect to t of the difference of (2) and (1) is greater than or equal to zero iff

$$c_2\lambda_2\bar{F}_1(t) - c_2f_1(t) \geq 0$$

iff

$$\lambda_2 \geq \frac{f_1(t)}{\bar{F}_1(t)} \quad .$$

$F_1(\cdot)$ is DFR by hypothesis and, hence, $f_1(t)/\bar{F}_1(t)$ is decreasing in t. There are three cases to consider:

a) $\dfrac{f_1(t)}{\bar{F}_1(t)} \geq \lambda_2$ for all t

b) $\dfrac{f_1(t)}{\bar{F}_1(t)} < \lambda_2$ for all t

c) $\dfrac{f_1(0)}{\bar{F}_1(0)} \geq \lambda_2$ and for all $\hat{t} > t_0$, $\dfrac{f_1(\hat{t})}{\bar{F}_1(\hat{t})} < \lambda_2$.

In case a) the partial with respect to t of equation (2) minus equation (1) is never positive and, hence, no crossing point exists (except in the case of strict equality). In case b) the difference in strictly increasing and, hence, exactly one change of sign occurs.

In case c) the difference is at first decreasing and then after a certain time (say t_0) strictly increasing in t. Thus, there exists a unique point s where the two equations (1) and (2) cross. ⊠

Using the results of the two previous theorems along with Theorem 5.2 (and its proof) conditions which insure a replace-repair policy is optimal can be derived. It is important to remember that if no cross over point (as defined in the previous two theorems) exists, then the optimal policy is the repair-only policy, a degenerate subset of the repair-replace policies. For the following theorems it will be assumed that a cross over point, or switching time, exists, is within the warranty period W, and occurs s units of time from the end of the warranty.

<u>Theorem 6.4</u>. If $c_1 > c_2$ and

$$dF_1(x) < dF_2(x) \qquad \forall x \in [0,W]$$

then a replace-repair policy is optimal.

<u>Proof</u>. By Theorem 6.2 the optimal policy ends in a replace-repair type policy. If it can be shown that a policy of the form repair-replace-repair is always more expensive than a replace-replace-repair (equivalent to replace-repair) policy then the theorem will be proved.

Consider the two policies depicted below.

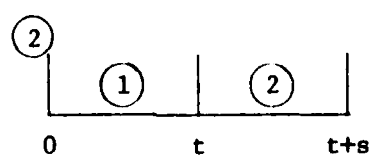


Figure 6.3. A Replace-Repair-Replace Policy (A)

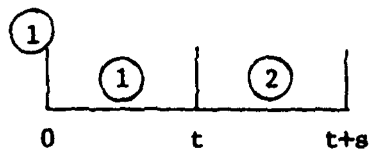


Figure 6.4. A Replace-Repair Policy (B)

In Policy A (Figure 6.3) one repairs if there is $s + t$ time units left in the warranty, replaces if there is less than $t + s$ and greater than or equal to s time units left, and repairs if there is less than s units left. Policy B (Figure 6.4) is identical except that replacement occurs at time $t + s$.

Using the notation of Example 6.1, the total expected cost of Policy A is

$$c_2 + \int_0^{t+s} V(t + s - x)dF_2(x)$$

and the total expected cost of Policy B is

$$c_1 + \int_0^{t+s} V(t + s - x)dF_1(x) .$$

Taking the difference (A - B) and then differentiating with respect to t yields

$$\frac{\partial}{\partial t} [A - B] = V(0)(dF_2(t + s) - dF_1(t + s))$$

$$+ \int_0^{t+s} \frac{\partial}{\partial t} [V(t + s - x)][dF_2(x) - dF_1(x)]$$

which is strictly increasing since $dF_2(x) > dF_1(x)$.

For $t = 0$ Policy A has a cost that is at least that of Policy B by definition of s. By the proof of Theorem 6.2 for some sufficiently small positive t Policy A is more costly than Policy B. By the above result the difference in cost between Policy A and Policy B is strictly increasing in t. Thus, Policy A can never be better than Policy B and, hence, a replace-repair policy is optimal. ☒

A similar proof will be used in the following related theorem.

Theorem 6.5. If $c_1 > c_2$, $F_2(\cdot)$ is an exponential distribution and $F_1(\cdot)$ is DFR with density $f_1(\cdot)$ then a replace-repair policy is optimal.

Proof. By Theorem 6.3 the optimal policy ends in a replace-repair type policy. By the proof of Theorem 5.2

$$\left.\frac{\partial V}{\partial t}\right|_{t+s} = c_2\lambda_2 \qquad \text{for Policy A (Figure 6.3)} .$$

For Policy B (by hypothesis $F_1(\cdot)$ has a density $f_1(\cdot)$)

$$\left.\frac{\partial}{\partial t} V(t)\right|_{t+s} = \frac{\partial}{\partial t}\int_0^{t+s} V(t + s - x)f_1(x)dx$$

$$= V(0)f_1(t+s) + \int_0^{t+T} \frac{\partial}{\partial t} V(t + s - x)f_1(x)dx$$

$$= V(0)f_1(t + s) + \int_0^t \frac{\partial}{\partial t} V(t + s - x)f_1(x)dx$$

$$+ \int_t^{t+s} \frac{\partial}{\partial t} V(t + s - x)f_1(x)dx \ .$$

By hypothesis $c_1 > c_2$. Thus, by Theorem 6.1 $V(0) = c_2$. By the definition of s and the hypothesis that $F_2(\cdot)$ is exponential

$$V(x) = c_2\lambda_2 x \qquad \forall x \in [0,s) \ .$$

Hence, for Policy B

$$\left.\frac{\partial V(t)}{\partial t}\right|_{t+s} = c_2 f_1(t + s) + \int_0^t \frac{\partial}{\partial t} V(t + s - x)f_1(x)$$

$$+ c_2\lambda_2(F_1(s + t) - F_1(t)) \ .$$

$V(x)$ is continuous since both $F_1(\cdot)$ and $F_2(\cdot)$ have densities. At time s

$$\left.\frac{\partial V(x)}{\partial x}\right|_s < c_2\lambda_2$$

and, in fact, if it is optimal to replace at some point $x_0 > s$, then

$$\left.\frac{\partial V(x)}{\partial x}\right|_{x_0} \leq c_2\lambda_c \ .$$

This can be seen either by a recursion argument on the derivative (identical to the rest of this proof) or by noting that if

$$\left.\frac{\partial V(x)}{\partial x}\right|_{x_0 > s} > c_2\lambda_2$$

then by the continuity of $V(x)$ and Theorem 5.2 it would be cheaper to repair at time x_0 (a contradiction).

Using this bound

$$\begin{aligned}
\left.\frac{\partial V(t)}{\partial t}\right|_{s+t} &= c_2 f_1(t + T) + \int_0^t \frac{\partial}{\partial t} V(t + s - x) f_1(x)dx \\
&\qquad + c_2\lambda_2(F_1(s + t) - F_1(t)) \\
&\le c_2 f_1(t + s) + \int_0^t c_2\lambda_2 f_1(x)dx \\
&\qquad + c_2\lambda_2(F_1(s + t) - F_1(t)) \\
&= c_2 f_1(t + s) + c_2\lambda_2 F_1(s + t) .
\end{aligned}$$

By the results of Theorem 6.3, the definition of s and the assumption that $F_1(\cdot)$ is DFR,

$$\frac{f_1(t+s)}{\bar{F}_1(s+t)} < \lambda_2 \qquad \text{or ,}$$

$$f_1(t+s) < \lambda_2 - \lambda_2 F_1(s + t) .$$

Substituting in (for Policy B)

$$\frac{\partial V(t)}{\partial t} < c_2\lambda_2 - c_2\lambda_2F_1(s + t) + c_2\lambda_2F_1(s + t)$$

$$= c_2\lambda_2 .$$

Thus, the difference in cost between Policy A and Policy B is strictly increasing in t, and by the same argument used in Theorem 6.4 a replace-repair policy is optimal. ☒

Using the above results the optimality of the exponential examples of Section 5.2 can be proved.

Lemma 6.6. If $F_1(\cdot)$ and $F_2(\cdot)$ are both distributed exponentially with parameters λ_1 and λ_2 and costs c_1 and c_2 respectively with $c_1 > c_2$ and $\lambda_1 < \lambda_2$ then the optimal policy is a replace-repair policy.

Proof. Exponential distributions are subsets of DFR distributions. Thus, Theorem 6.5 applies. If $\lambda_1 < \lambda_2$ then condition b) of Theorem 6.3 applies, exactly one switching point exists, and a replace-repair policy with nonzero s is optimal. ☒

Using the results of this chapter an alternate method for finding s in the exponential example can be used.

Example 6.2. If F_1 and F_2 are exponential distributions, then the first and only switching time occurs at the first (and only) zero crossing of

$$c_2 - c_1 + \int_0^t c_2(1 + M_2(t - x))[dF_2(x) - dF_1(x)]$$

$$= c_2 - c_1 + \int_0^t c_2(1 + \lambda_2(t - x))\left[\lambda_2 e^{-\lambda_2 x} - \lambda_1 e^{-\lambda_1 x}\right]dx$$

$$= c_2 - c_1 + c_2\lambda_2 t - c_2\left(1 - e^{-\lambda_1 t}\right) - c_2\lambda_2 t\left(1 - e^{-\lambda_1 t}\right)$$

$$+ \int_0^t c_2\lambda_2 \, x\lambda_1 e^{-\lambda_1 x} \, dx$$

$$= c_2\lambda_2\mu_1 - c_1 + c_2 e^{-\lambda_1 t} - c_2\lambda_2\mu_1 e^{-\lambda_1 t} \quad .$$

Solving for the point s where this quantity equals zero

$$c_1 - c_2\lambda_2\mu_1 = (c_2 - c_2\lambda_2\mu_1)e^{-\lambda_1 s}$$

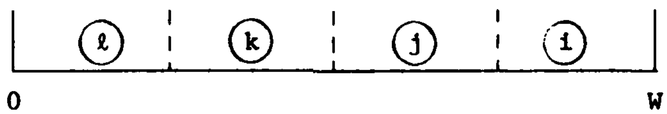
or

$$s = \frac{1}{\lambda_1} \ln\left[\frac{c_2\mu_1 - c_1\mu_2}{c_2\mu_1 - c_2\mu_2}\right] \quad ,$$

the same value derived in Section 5.2.

Two concluding notes on optimal repair policies seem appropriate here. First of all, as was stated in Chapter 5, limiting the analysis to the standard warranty does not exclude (W,T) warranty policies. The statements and theorems of Chapter 6 (with the obvious exception of Theorem 6.1) are also applicable to (W,T) policies.

Secondly, if the manufacturer has multiple options (i.e., $F_1(\cdot)$, $F_2(\cdot)$, ..., $F_n(\cdot)$) that are <u>all</u> distributed exponentially, then the results of Chapters 5 and 6 can be used to show that an optimal repair policy has the form



where $c_i\lambda_i > c_j\lambda_j > c_k\lambda_k > c_\ell\lambda_\ell \cdots$. The proof is not included here because an excellent proof (albeit very different in both approach and technique) can be found in [10].

CHAPTER 7

CONCLUSIONS AND FUTURE RESEARCH

The major results of this paper fit into three catagories: defining warranty policies, mathematical analysis of these policies, and consideration of the manufacturer's option of repairing an item in lieu of replacing it. Six different warranty policies were defined and analyzed. The standard warranty, the renewing warranty and the pro rata warranty are all common policies found nation wide. The pro rata with rebate and pro rata with delay warranty policies (see Chapter 3) are beginning to become popular and have appeared in recent literature [4, 5, 13]. The (W,T) warranty policy is new and has not appeared in any published literature (to this author's knowledge). However, it is currently being used by such major companies as Texas Instruments.

Each of the above policies was analyzed from both the manufacturer's point of view (profit per customer per unit time) and from the consumer's point of view (cost per unit time) over both finite and infinite time horizons. This analysis was performed primarily by the use of renewal equations.

When the manufacturer is faced with the option of repairing or replacing an item under warranty the question of optimal repair policies arises. One repair policy, the replace-repair policy (replace if there is greater than s time units left in the warranty and repair otherwise), was shown to be optimal whenever the repaired item had an exponential life length distribution and the replaced or new item had a DFR distribution.

There are many areas for future research. From the econometric side one could assume consumer demand as a function of both price and warranty to arrive at the optimal warranty that should be offered. Or, discounting could be considered by folding the discount rate into the failure distribution to arrive at a terminating renewal process (Feller [11] calls the analysis of this process "trite"). From the sociologic point of view the percent of eligible consumers who actually use their warranties could be included in the analysis of Chapters 3 and 4.

An intriguing question to those interested in reliability is what other conditions can be derived which insure a replace-repair policy is optimal. For instance, it might be conjectured, due to the symmetry that frequently exists in these processes, that if the replaced item had an exponential life length distribution and the repaired item an IFR distribution, then the optimal repair policy would also be a replace-repair policy.

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A warranty is a contractual obligation incurred by a manufacturer or vendor in connection with the sale of an item or service. A typical warranty specifies that the manufacturer agrees to remedy certain defects or failures, in the commodity sold, within a predetermined time span and at a particular cost to the consumer.

This study considers full and limited warranty policies from both the consumers' and manufacturers' point of view. The consumers' expected cost over

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20. WARRANTY POLICIES: CONSUMER VALUE VS. MANUFACTURER COST (Continued)

a given time horizon is calculated for the various policies. Using these values, the additional cost the consumer should be willing to pay for a given warranty is found. To the manufacturer, profit is the most important consideration. In view of this, the expected profit per customer per unit time is calculated for each policy so that the manufacturer can set prices accordingly.

The analysis is extended to include the manufacturers' option of repairing an item in lieu of replacing it. Conditions are derived which ensure the existence of an optional "switching time" or time after which the manufacturer should repair instead of replace.

Many of the results are derived by the use of renewal equations and delayed renewal equations. In each case a probability distribution function is assumed to govern the life length of the item.

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